MA 341 first midterm review problems
Version as of September 19th.

The first midterm will be in class on Monday, September 27th. Justify your answers. No notes or electronic devices allowed. Most of the problems on the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Find a natural number N such that if $|a-5| < \varepsilon < 1$ then $|a^2 - 25| < N\varepsilon$.

2. Find a number c such that $0 < c < 1$ and if $|a-5| < \varepsilon < 1$ then $|\sqrt{a} - \sqrt{5}| < c\varepsilon$.

3. For $n \geq 1$, define the sequence
$$a_n = 1 + \frac{1}{\sqrt[3]{2}} + \cdots + \frac{1}{\sqrt[3]{n}}.$$
Find a number C such that the sequence $b_n = a_n - C(n+1)^{2/3}$ has a limit L. Do not try to find L, but instead find a number M such that L is inside the interval $[M, M+1]$.

4. For each of the following sequences, find the limit L of the sequence. Then find N such that $|a_n - L| < 0.1$ for $n > N$.
   (a)
$$a_n = \frac{1}{n^2}$$
   (b)
$$a_n = \frac{\sin n - \cos(n^2) + \sin^2(n^3)}{n^4}$$
   (c)
$$a_n = \frac{2n^3 + \cos n + 2^{-n}}{n^3 + 4}$$

5. Let a, b and ε be real numbers such that $|a-3| < \varepsilon$, $|a-b| < 2\varepsilon$, and $|b-3.5| < \varepsilon$. Use the triangle inequality to find a number c, as large as possible, such that $\varepsilon > c$.

6. Let $a_n = 3^n/n!$
   (a) Find N such that $a_{n+1} \leq a_n$ for $n > N$.
   (b) Find N such that $a_{n+1} \leq a_n/5$ for $n > N$.
   (c) Find the largest term in the sequence. Is more than one term tied for largest?
   (d) Do the same problem with $a_n = 100^n/n!$

7. (a) Beginning with the formula
$$1 + a + \cdots a^n = \frac{1}{1-a} + e_n, \qquad e_n = -\frac{a^{n+1}}{1-a},$$

substitute $a = -u^2$, and integrate from 0 to $1/\sqrt{3}$ using $\int_0^b (1+u^2)^{-1} du = \arctan(b)$ and solve the resulting equation for π. Prove that the remainder, obtained from integrating e_n, goes to 0 as $n \to \infty$. Hence, find natural numbers $A,\ B,\ C, \ldots$ such that

$$\pi = A\sqrt{3}\left(1 - \frac{1}{B \cdot C} + \frac{1}{D \cdot C^2} - \frac{1}{E \cdot C^3} + \cdots\right)$$

(Incidentally, this formula was used in 1719 to compute π up to 112 digits: see Section I.4 of *Analysis by Its History* by Hairer and Wanner.)

8. Find two cluster points of each of the following sequences, and, for each cluster point, find a subsequence converging to it.

   (a)
   $$a_n = (-1)^n$$

   (b)
   $$a_n = (-1)^n - (-2)^{-n}$$

   (c)
   $$a_n = \sin(n\pi/10)$$

   (d)
   $$a_n = \cos\left(n\pi + \frac{\pi}{n}\right)$$

   Hint: $\cos(x + y) = \cos x \cos y - \sin x \sin y$

9. Decide if each of the following series is absolutely convergent, conditionally convergent, or divergent.

   (a)
   $$\sum_{n=2}^{\infty} (-1)^n \frac{n^5}{n^6 - 1}$$

   (b)
   $$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

   Hint: $\lim(1 + \frac{1}{n})^n = e$.

   (c)
   $$\sum_{n=1}^{\infty} \frac{5^n n!}{n^n}$$

10. Show that if $\sum a_n$ converges absolutely, then $\sum a_n^8$ does too. Find an example showing this does not hold for conditional convergence.

11. For each of the following sets, find the sup, inf, max, and min, if they exist.

    (a) $\{n^2 - 9n \colon n = 1,\ 2,\ 3, \ldots\}$

(b) $\{-n^2 + 10n + \sin n \colon n = 1,\ 2,\ 3, \dots\}$

(c) $\{\sin n\pi/2 \colon n = 1,\ 2,\ 3, \dots\}$

(d) $\{\sin n\pi/1000 \colon n = 1,\ 2,\ 3, \dots\}$

(e) $\{\sin n\pi/7 \colon n = 1,\ 2,\ 3, \dots\}$

(f) $\{(-1)^n n^{-1} \sin n \colon n = 1,\ 2,\ 3, \dots\}$

For this part and the next one you may use the fact that $.8 < \sin 1 < .9 < \sin 2$

(g) $\{n^{-1} \sin^2 n \colon n = 1,\ 2,\ 3, \dots\}$

12. Find the radius of convergence of each of the following power series

(a)

$$\sum_{n=1}^{\infty} \frac{x^{2n}}{n!}$$

(b)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}x^{3n}}{8^n}$$

(c)

$$\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$