MA 341 second midterm review problems
Version as of November 4th.

The second midterm will be in class on Friday, November 5th. Justify your answers. No notes or electronic devices allowed. Most of the problems on the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Let

$$f(x) = \begin{cases} \cos x, & x < 1, \\ 4, & x = 1, \\ -(x-3)^2, & x > 1. \end{cases}$$

   Determine whether sup, inf, max, and min of $f(x)$ exist on each of the following intervals, and determine their values if they do.

   (a) $[0, 5]$

   (b) $(0, 5)$

   (c) $[4, \infty)$

   (d) $(-\infty, 0)$

2. Find numbers a and b such that for all real c we have

$$0 < a \leq \int_0^2 \frac{x^2}{4 + \cos(cx)} dx \leq b < 1.$$

3. In each of the following problems, given $\varepsilon > 0$, find $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ when $|x - x_0| < \delta$.

   (a)

$$f(x) = \frac{2}{1 + 3x^2}, \qquad x_0 = 0$$

   (b)

$$f(x) = \int_0^2 \frac{3}{1 + 4xt^4}\, dt, \qquad x_0 = 0$$

   (c)

$$f(x) = \sqrt{x^2 + 1}, \qquad x_0 = 2$$

4. Let $f\colon [0, \infty) \to \mathbb{R}$ be continuous and such that

$$\lim_{x \to \infty} f(x) = 37.$$

   Prove that f is bounded.

5. Let $f\colon (-10, 1) \to \mathbb{R}$ be continuous, such that

$$\lim_{x \to 1} f(x) = +\infty,$$

   and such that f is decreasing on $(-10, -5)$. Show that f has a minimum on $(-10, 1)$.

6. Suppose that $f'(x) \geq 10$ for all x in $[0, 5]$. Find $a > 0$, as large as possible, such that there is guaranteed to be an interval I of length a contained in $[0, 5]$ such that $|f| \geq 2$ on I. Justify your answer using the mean value theorem.

7. Let $a \neq 0$ be given, and let $n \geq 2$ be an even integer. Prove that $x = 0$ is the only solution to $x^n + a^n = (x + a)^n$.

   Hint: Argue by contradiction and use the mean value theorem.

8. Let $f\colon [0, 1] \to \mathbb{R}$ be a continuous function such that $-2 \leq f(x) \leq 4$ for all x in $[0, 1]$. Let $g\colon [0, 1] \to \mathbb{R}$ be a continuous function such that $\inf_{[0,1]} g(x) = -2$ and $\sup_{[0,1]} g(x) = 4$. Prove there is x_0 in $[0, 1]$ such that $f(x_0) = g(x_0)$.

9. Find a number $c < 1$ such that for all real a we have

$$\int_0^{10} \frac{\cos(ax)}{1 + e^x} dx \leq c.$$