Sums of independent random variables

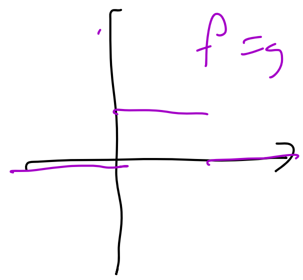
Density function of a sum given by convolution:

If f is pdf of X, g of Y, indep. then pdf of sum is

$$h(z) = \int_{-\infty}^{\infty} f(z-y)\, g(y)\, dy$$

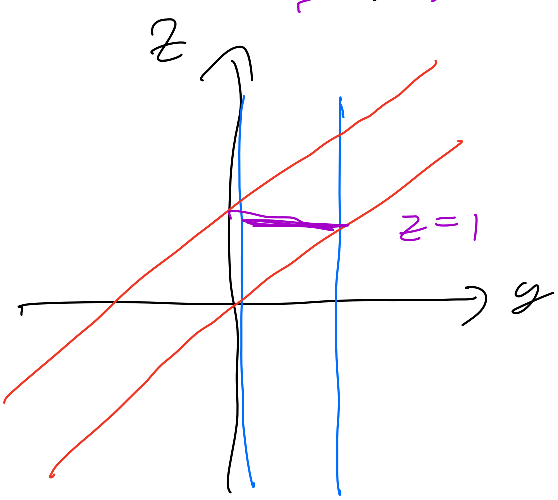
Example Uniform density on $[0,1]$

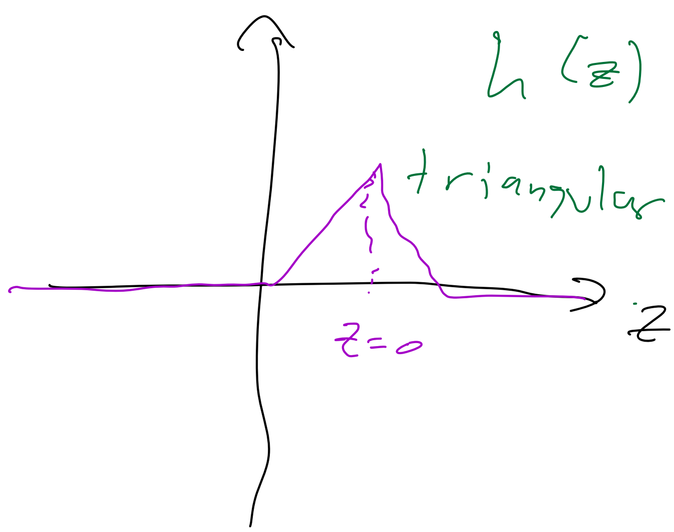
$$f(x) = g(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$



$$h(z) = \int_0^1 f(z-y)\, dy = \int_{\substack{0 \le y \le 1 \\ 0 \le z-y \le 1}} 1 \, dy$$

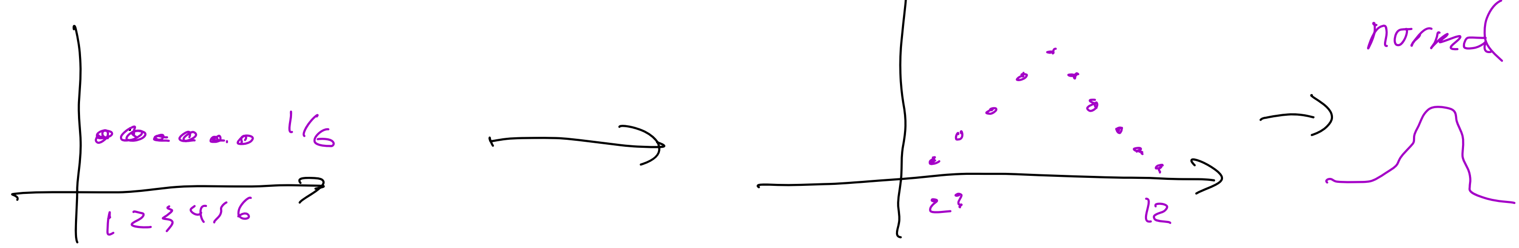
$$= \text{length of } \{ y : 0 \le y \le 1,\ 0 \le z-y \le 1 \}$$





Similar rolling two dice:

each die is   sum is



<u>General rule</u>: Convolution is "nicer" than inputs, more gradual, eventually repeated convolving results in normal distribution

<u>Central limit theorem</u> makes this precise

This is Ch 8, we will just do pieces of 7 & 8.

Use <u>Moment generating functions</u> to analyze sums.

<u>Def</u> For any random variable X, define

$M_X(t) = E[e^{tX}]$. This generates moments

by taking $M_X^{(k)}(0)$. Concretely

leading taylor coeffs

0) $M_X(0) = E[e^{0X}] = E[1] = 1$

1) $M_X'(0) = E[Xe^{0X}] = E[X]$

2) $M_X''(0) = E[X^2 e^{0X}] = E[X^2]$

moments of X

If X & Y are independent:

$$M_{X+Y}(t) = E\left[e^{t(X+Y)}\right] = E\left[e^{tX} e^{tY}\right]$$

$$= E\left[e^{tX}\right] E\left[e^{tY}\right] = M_X(t) M_Y(t)$$

Keypoint: Moment generating function replaces convolution by product. (homomorphism in algebra)

Central Limit Theorem Recall, if $Y_1, \dots, Y_n, \dots$ are independent Bernoulli, i.e. biased coin flip with probability p,

$$P\left(a \le \frac{Y_1 + \cdots + Y_n - np}{\sqrt{np(1-p)}} \le b\right) \to \Phi(b) - \Phi(a)$$

standard normal

rewrite as

average

individual mean & st. dev.

$$P\left(a \le \frac{\frac{Y_1 + \cdots + Y_n}{n} - p}{\sqrt{p(1-p)}} \sqrt{n} \le b\right) \to \Phi(b) - \Phi(a)$$

CLT says $P\left(a \le \frac{\frac{Y_1 + \cdots + Y_n}{n} - \mu}{\sigma} \sqrt{n} \le b\right) \to \Phi(b) - \Phi(a)$

for any independent, identically distributed (i.i.d.) random variables of mean μ, variance σ^2

Can derive Law of Large numbers, which states that we converge to the mean:

bands getting tighter as n grows

$$P\left(\frac{a\sigma}{\sqrt{n}} \le \frac{Y_1 + \cdots + Y_n}{n} - \mu \le \frac{b\sigma}{\sqrt{n}}\right) \to \Phi(b) - \Phi(a)$$

take $a = -n^{1/4}$, $b = n^{1/4}$

$$P\left(\frac{-\sigma}{n^{1/4}} \le \frac{Y_1 + \cdots + Y_n}{n} - \mu \le \frac{\sigma}{n^{1/4}}\right) \to \Phi(+\infty) - \Phi(-\infty) = 1$$

$$P\left(\frac{Y_1 + \cdots + Y_n}{n} = \mu\right) \to 1$$

Law of large numbers LLN

LLN also more generally for distinctly distributed variables,

Two biggest theorems in Probability.

Derive CLT using moment generating functions

Looking at $\dfrac{\left(\frac{Y_1 + \cdots + Y_n}{n} - \mu\right)\sqrt{n}}{\sigma}$

want to show converges to standard normal

Start by substitution: $X_j = \dfrac{Y_j - \mu}{\sigma}$

Then $X_1, \dots, X_n, \dots$ are indep, i.i.d.,

have mean 0 and variance 1.

Want $\dfrac{\frac{X_1 + \dots + X_n}{n} - 0}{1} \sqrt{n} = \dfrac{X_1 + \dots + X_n}{\sqrt{n}} \to$ standard normal

$M_{X+Y}(t) = E[e^{t(X+Y)}]$
$= E[e^{tX}]E[e^{tY}]$

Take moment generating function

major property of M

$$M_{\frac{X_1 + \dots + X_n}{\sqrt{n}}}(t) = M_{\frac{X_1}{\sqrt{n}}}(t)\, M_{\frac{X_2}{\sqrt{n}}}(t) \dots M_{\frac{X_n}{\sqrt{n}}}(t)$$

def of M

$$= E\left[e^{\frac{X_1 t}{\sqrt{n}}}\right] E\left[e^{\frac{X_2 t}{\sqrt{n}}}\right] \dots E\left[e^{\frac{X_n t}{\sqrt{n}}}\right]$$

identically distrib

$$= E\left[e^{\frac{X_1 t}{\sqrt{n}}}\right]^n = M_X\left(\frac{t}{\sqrt{n}}\right)^n$$

Taylor expand

$$= \left(M_X(0) + \frac{t}{\sqrt{n}} M_X'(0) + \frac{t^2}{2n} M_X''(0) + \dots\right)^n$$

mean 0 variance 1

$$= \left(1 + \frac{t}{\sqrt{n}} E[X] + \frac{t^2}{2n} E[X^2] + \dots\right)^n$$

$(1+\frac{s}{n})^n \to e$

$$= \left(1 + \frac{t^2}{2n} + \dots\right)^n \to e^{\frac{t^2}{2}}$$

Upshot $M_{\frac{X_1+\cdots+X_n}{\sqrt{n}}}(t) \longrightarrow e^{t^2/2}$

Next Show that standard normal Z has $M_Z(t) = e^{t^2/2}$

compute using def:

$$M_Z(t) = E[e^{tZ}] = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

complete the square

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} e^{tz} e^{-\frac{z^2}{2}} dz$$

u

$$= \frac{1}{\sqrt{2\pi}} e^{\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(z-t)^2}{2}} dz$$

du

standard normal integral

$\Phi(\infty) = 1$

$$= e^{\frac{t^2}{2}}$$

That was the central limit theorem.

- More practice with moment generating functions.

These are variations of Fourier transform / MA [428]

Laplace transform as in differential equations, etc.

Fourier transform is $\hat{f}(p) = \int_{-\infty}^{\infty} e^{-ixp} f(x)\, dx$

Laplace is $\mathcal{L}[f](s) = \int_0^{\infty} e^{-st} f(t)\, dt$

Moment is $M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

Most gen'lly Fourier analysis relates f and its transforms

Examples for moments

1) Bernoulli $M_x(t) = E[e^{tx}] = \sum_{j=0}^{1} e^{t\cdot 0} p(0) + e^{t\cdot 1} p(1)$

$= 1\cdot(1-p) + e^t p = 1-p+e^t p$

2) Normal $M_x(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}} dx = e^{t^2/2}$

3) Exponential: $M_x(t) = \int_0^\infty e^{tx} e^{-x\lambda} \lambda\, dx$

Varied looking functions

$= \frac{\lambda}{t-\lambda} e^{(t-\lambda)x}\Big|_{x=0}^{x=\infty} = \frac{\lambda}{\lambda - t}$, provided $t < \lambda$

4) Geometric: $p(j) = p(1-p)^{j-1}$

$M_x(t) = \sum_{j=1}^{\infty} e^{-tj} p(1-p)^{j-1} = \frac{p}{1-p} \sum_{j=1}^{\infty} \left(e^{-t}(1-p)\right)^j$

$= \frac{p}{1-p} \frac{e^{-t}(1-p)}{1-e^{-t}(1-p)} = \frac{p(1-p)}{(1-p)e^t - (1-p)^2} = \frac{p}{e^t - 1 + p}$

Check $M(0) = 1$ always

$M'(0) = E[X]$, geometric

geometric $M'(t) = \left(\frac{p}{e^t - 1 + p}\right)' = \frac{p}{(e^t - 1 + p)^2} e^t$

$M'(0) = \frac{p}{p^2} = \frac{1}{p}$

$M''(0) = E[X^2]$

$M''(t) = \frac{-2p}{(e^t - 1 + p)^3} + \frac{p}{(e^t - 1 + p)^2} e^t$

$$M''(0) = \frac{-2p}{p^3} + \frac{1}{p} = \frac{p-2}{p^2}$$

gives us another way to compute variance, without sum formulas beyond geometric sum

Upshot May use $M_X(t)$ to compute, $E[X]$, $Var(X)$, etc., Sometimes simpler

Fourier inversion:

Let $\hat{f}(p) = \int_{-\infty}^{\infty} e^{-itp} f(t)\,dt$

Then $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itp} \hat{f}(p)\,dp$