CLAIRAUT'S THEOREM

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Clairaut's theorem says that if the second partial derivatives of a function are continuous, then the order of differentiation is immaterial.

Theorem. *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}$ *have all partial derivatives up to second order continuous near* (a,b)*. Then* $\partial_x\partial_y f(a,b) = \partial_y\partial_x f(a,b)$*.*

Proof. By definition,

$$\begin{aligned}\partial_x\partial_y f(a,b) &= \lim_{h\to 0}\frac{\partial_y f(a+h,b) - \partial_y f(a,b)}{h}\\ &= \lim_{h\to 0}\lim_{k\to 0}\frac{f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b)}{hk}.\end{aligned}$$

Apply the mean value theorem to the function $g(t) = f(a+t,b+k) - f(a+t,b)$ on the interval $[0,h]$ to get that there is a^* between a and $a+h$ such that

$$f(a+h,b+k) - f(a,b+k) - f(a+h,b) + f(a,b) = \left[\partial_x f(a^*,b+k) - \partial_x f(a^*,b)\right]h,$$

so that

$$\partial_x\partial_y f(a,b) = \lim_{h\to 0}\lim_{k\to 0}\frac{\partial_x f(a^*,b+k) - \partial_x f(a^*,b)}{k}.$$

By definition,

$$\lim_{k\to 0}\frac{\partial_x f(a^*,b+k) - \partial_x f(a^*,b)}{k} = \partial_y\partial_x f(a^*,b),$$

giving

$$\partial_x\partial_y f(a,b) = \lim_{h\to 0}\partial_y\partial_x f(a^*,b).$$

Since $a^* \to a$ as $h \to 0$, and since $\partial_y\partial_x f$ is continuous, that gives

$$\partial_x\partial_y f(a,b) = \partial_y\partial_x f(a,b).$$

□

The same applies to functions of more than two variables, because to interchange the order of differentiation we only ever have to consider two variables at a time.

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