DIFFERENTIAL FORMS

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1. Introduction

Let $\gamma\colon [a,b] \to \mathbb{R}^n$ be C^∞, one-to-one, and have nonvanishing derivative, and let $F_1, \dots, F_n$ be C^∞ functions on $\mathbb{R}^n$. We define

$$\int_{\gamma([a,b])} F_1 dx^1 + \cdots + F_n dx^n = \int_a^b F(\gamma(t))^T \dot\gamma(t) dt,$$

where $F(\gamma(t))^T$ is the row vector $(F_1(\gamma(t), \dots, F_n(\gamma(t))$ and $\dot\gamma(t)$ is the column vector $\frac{d}{dt}\gamma(t)$. This integral gives the work done by the force F moving a particle along the length of the path $\gamma([a,b])$.

To understand better the object being integrated on the left side, let $\sigma\colon \mathbb{R} \to \mathbb{R}^m$ be another C^∞ curve, and let $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ be a C^∞ mapping, such that $\gamma = \varphi \circ \sigma$. Then write

$$\int_a^b F(\gamma(t))^T \dot\gamma(t) dt = \int_a^b F(\gamma(t))^T D\varphi(\sigma(t)) \dot\sigma(t) dt = \int_a^b G(\sigma(t))^T \dot\sigma(t) dt,$$

where $G(u) = D\varphi(u)^T F(\varphi(u))$, so that $F_1 dx^1 + \cdots F_n dx^n$ corresponds under φ with $G_1 du^1 + \cdots + G_m du^m$. Thus F transforms like a row vector, i.e. like a linear mapping from vectors to numbers. In the next section we define $F_1 dx^1 + \cdots F_n dx^n$ accordingly.

2. One-forms

2.1. **Definition.** A *vector field* on $\mathbb{R}^n$ is a first order differential operator $V = V^1 \partial_{x^1} + \cdots + V^n \partial_{x^n}$, where the V^j are C^∞ functions of x. Vector fields map the space of functions $C^\infty(\mathbb{R}^n)$ to itself: $Vf = \sum_{j=1}^n V^j \partial_{x^j} f$.

A *one-form* on $\mathbb{R}^n$, $\alpha = F_1 dx^1 + \cdots + F_n dx^n$ where the F_j are C^∞ functions of x, maps vector fields to functions. We write $\langle \alpha, V \rangle = \sum_{j=1}^n F_j V^j$.

The *exterior derivative* operator d takes functions to one-forms by $df = \sum_{j=1}^n \partial_{x^j} f dx^j$. Thus $\langle df, V \rangle = Vf$. Note that this is consisent with the previous definition of dx^j because $Vx^j = V^j$.

If $\varphi\colon \mathbb{R}^m \to \mathbb{R}^n$ is a C^∞ mapping, the *pullback* operator φ^* takes functions and one-forms on $\mathbb{R}^n$ to functions and one-forms on $\mathbb{R}^m$ by $\varphi^* f = f \circ \varphi$ and $\varphi^*(\sum_{j=1}^n F_j dx^j) = \sum_{j=1}^n (\varphi^* F_j)(d\varphi^* x_j)$. Thus pullback distributes over sums and product and commutes with exterior differentiation.

As an example, if $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\varphi(u,v) = (u+v, uv)$, then

$$\varphi^* x = u + v, \quad \varphi^* y dx = uvd(u+v) = uvdu + uvdv, \quad \varphi^* dy = d(uv) = vdu + udv.$$

2.2. **Integration of one-forms.** We define

$$\int_{\gamma[a,b]} \alpha = \int_a^b \langle \alpha, \dot\gamma \rangle,$$

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and observe that $\gamma^*\alpha = \langle \alpha, \dot\gamma \rangle dt$, so that we may also write

$$\int_{\gamma[a,b]} \alpha = \int_{[a,b]} \gamma^*\alpha.$$

In the above integrals we pay attention to the orientation; the integral of a differential form depends not only on the set over which it is integrated, but also on the orientation of that set, in this case from $\gamma(a)$ to $\gamma(b)$ on the left sides of the equations and from a to b on the right sides.

Hence, by the fundamental theorem of calculus,

$$\int_{\gamma[a,b]} df = \int_a^b \partial_{x^j} f \dot x^j dt = f(\gamma(b)) - f(\gamma(a)).$$

We write this as

$$\int_C df = \int_{\partial C} f, \tag{1}$$

where C is the curve $\gamma[a,b]$ oriented from $\gamma(a)$ to $\gamma(b)$, and ∂C is the boundary of C with the corresponding orientation, namely the two point set $\{\gamma(a), \gamma(b)\}$ with the first point oriented by $-$ and the second by $+$. We will generalize this formula to integrals of any dimension.

3. Higher order forms

3.1. **Definitions.** The simplest two-dimensional analogue of (1) is obtained by integrating a one around a rectangle. Let a and b be positive numbers, and let C_1 be the segment from $(0,0)$ to $(a,0)$, C_2 the segment from $(a,0)$ to (a,b), C_3 the segment from (a,b) to $(0,b)$ and C_4 the segment from $(0,b)$ to $(0,0)$. Let C be the concatenation of C_1, C_2, C_3, C_4, so the whole perimeter of the rectangle, and let R be the interior of the rectangle. Then

$$\int_C F_1 dx^1 = \int_{C_1} F_1 dx^1 + \int_{C_3} F_1 dx^1 = \int_0^a (F_1(t,0) - F_1(t,b))dt = -\int_0^a\int_0^b \partial_2 F_1(t,s) ds dt = -\int_R \partial_2 F_1.$$

Similarly,

$$\int_C F_2 dx^2 = \int_R \partial_1 F_2,$$

and adding these gives *Green's formula* for a rectangle:

$$\int_R (\partial_1 F_2 - \partial_2 F_1) = \int_C F_1 dx^1 + F_2 dx^2. \tag{2}$$

To compare this formula with (1) we need to introduce a notion of orientation for double integrals. To see what is needed, switch the two variables, putting $\varphi(u^1, u^2) = (u^2, u^1)$. Then

$$\varphi^*(F_1 dx^1 + F_2 dx^2) = G_1 du^2 + G_2 du^1,$$

with $G_j(u^1, u^2) = F_j(u^2, u^1)$, so

$$\int_C F_1 dx^1 + F_2 dx^2 = \int_{\varphi^{-1}(C)} G_1 du^1 + G_2 du^2.$$

And

$$\partial_1 F_2 = \partial_2 G_1, \qquad \partial_2 G_1 = \partial_1 F_2,$$

giving

$$-\int_{\varphi^{-1}(R)} (\partial_1 G_2 - \partial_2 G_1) = \int_{\varphi^{-1}(C)} G_1 du^1 + G_2 du^2.$$

Thus the sign in our formula flipped under this change of variables; this is because the orientation of the curve is reversed.

To keep track of this, we define a two-form $f(x,y)dx \wedge dy$ (the symbol $\wedge$ is called *wedge*) to be an *antisymmetric* mapping on two vector fields, i.e. when i and j are different we define basic two-forms by

$$\langle dx^i \wedge dx^j, (\partial_{x^k}, \partial_{x^\ell})\rangle = \begin{cases} 1, & \text{when } i = k \text{ and } j = \ell, \\ -1, & \text{when } i = \ell \text{ and } j = k, \\ 0, & \text{otherwise,} \end{cases}$$

and we put $dx^i \wedge dx^i = 0$. More generally, extending this linearly, we have

$$\langle f dx^i \wedge dx^j, (V, W)\rangle = f(V^i W^j - V^j W^i) = f \det \begin{bmatrix} V^i & W^i \\ V^j & W^j \end{bmatrix}.$$

General two-forms are mappings on two vector fields of the form

$$\sum_{1 \le i < j \le n} f_{ij}(x) dx^i \wedge dx^j \quad \text{or} \quad \sum_{i,j=1}^{n} g_{ij}(x) dx^i \wedge dx^j.$$

In the second case the terms with $i = j$ are all zero, and the terms with $j < i$ can be grouped with terms $i < j$ using $dx^i \wedge dx^j = -dx^j \wedge dx^i$, to rewrite in the form of the first case, i.e. $f_{ij} = g_{ij} - g_{ji}$.

Similarly we define basic three-forms

$$\langle dx^i \wedge dx^j \wedge dx^k, (U, V, W)\rangle = \det \begin{bmatrix} U^i & V^i & W^i \\ U^j & V^j & W^j \\ U^k & V^k & W^k, \end{bmatrix}.$$

General three-forms are mappings on three vector fields of the form

$$\sum_{1 \le i < j < k \le n} f_{ijk}(x) dx^i \wedge dx^j \wedge dx^k \quad \text{or} \quad \sum_{i,j,k=1}^{n} g_{ijk}(x) dx^i \wedge dx^j \wedge dx^k.$$

The same works for p-forms for any p, just with more elaborate notation. See Section 4.1 of Taylor's book [Ta]. Note that if we write a p-form as

$$\sum_{1 \le i_1 < \cdots < i_p \le n} f_{i_1 \ldots i_p}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

the number of terms in the sum is the binomial coefficient $\binom{n}{p}$. More specifically, we have 1 term when $p = 0$ (zero-forms are just functions), we have n terms when $p = 1$ (the same one-forms we discussed previously), then the number of terms increases with p for $p < n/2$ and decreases with p for $p > n/2$, until we have n terms again when $p = n - 1$ and 1 term when $p = n$. When $p > n$ the only p-form is the one which is identically zero (the determinant defining the basic p-form always vanishes then.)

3.2. **Wedge product and exterior derivative.** We combine a k-form and a p-form using the *wedge product*:

$$(f dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge (g dx^{j_1} \wedge \cdots \wedge dx^{j_p}) = fg dx^{i_1} \wedge \cdots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p},$$

and this product is defined to distribute over sums. For example, if $\varphi(u, v) = (uv, u^2 + v^2)$, then

$$\begin{aligned} \varphi^*(dx \wedge dy) &= d(\varphi^* x) \wedge d(\varphi^* y) = (udv + vdu) \wedge (2udu + 2vdv) \\ &= u2udv \wedge du + v2udu \wedge du + u2vdv \wedge dv + v2vdu \wedge dv \\ &= 2(v^2 - u^2) du \wedge dv. \end{aligned}$$

We extend the exterior derivative to p-forms by putting

$$d(f(x)dx^{i_1} \wedge \cdots \wedge dx^{i_p}) = df \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

and extending linearly. Thus

$$\begin{aligned} d(F_1 dx^1 + F_2 dx^2) &= dF_1 \wedge dx^1 + dF_2 \wedge dx^2 \\ &= (\partial_{x^1} F_1 dx^1 + \partial_{x^2} F_1 dx^2) \wedge dx^1 + (\partial_{x^1} F_2 dx^1 + \partial_{x^2} F_2 dx^2) \wedge dx^2 \\ &= (\partial_{x^1} F_2 - \partial_{x^2} F_1) dx^1 \wedge dx^2. \end{aligned} \tag{3}$$

3.3. **Integration of top order forms.** We first define the integral of an n-form α over an open set Ω in $\mathbb{R}^n$.

We already know how to do this when $n = 1$ and Ω is the interval (a, b). Then we write $\alpha = A(x)dx$ and put

$$\int_\Omega \alpha = \int_a^b A, \quad \text{or} \quad \int_\Omega \alpha = \int_b^a A = -\int_a^b A, \tag{4}$$

according to whether the interval is oriented from a to b or from b to a. We call the first case the *standard orientation* and that is what is meant when an orientation is not otherwise specified. An equivalent definition which generalizes nicely is that an *orientation* on Ω is given by choosing a nonvanishing one-form α_0 and declaring it to be *positive*. Then $\int_\Omega \alpha$ is defined by whichever formula in (4) makes $\int_\Omega \alpha_0 > 0$. For the standard orientation the simplest choice is $\alpha_0 = dx$.

If $x = \varphi(u)$ is a change of variables, then $\varphi^*\alpha = A(\varphi(u))\varphi'(u)du$. We say that φ is *orientation preserving* if $\varphi' > 0$ and *orientation reversing* if $\varphi' < 0$ (recall that φ' is nowhere vanishing by definition). Thus orientation-preserving changes of variables are the ones that preserve the positive one-forms.

These definitions generalize directly to n dimensions. We write can write a general n-form as

$$\alpha = A(x)dx^1 \wedge \cdots \wedge dx^n,$$

and an *orientation* on Ω is a choice of nonvanishing n-form α_0 which is declared to be *positive*. Then we define the integral of a general n-form α over Ω as whichever choice of

$$\int_\Omega \alpha = \int_\Omega A \quad \text{or} \quad \int_\Omega \alpha = -\int_\Omega A,$$

makes $\int_\Omega \alpha_0 > 0$. The *standard orientation* is given by $\alpha_0 = dx^1 \wedge \cdots \wedge dx^n$.

Combining these definitions with (2) and (3), we see that Green's formula for an open rectangle Ω and its boundary $\partial\Omega$ can be written as an analogue of (1):

$$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha,$$

where we use the standard orientation on Ω when $\partial\Omega$ is oriented counterclockwise and the reversed orientation on Ω when $\partial\Omega$ is oriented clockwise. (We will see soon how to specify the orientation on $\partial\Omega$ in terms of a nonvanishing one-form.)

If $x = \varphi(u)$ is a change of variables, then to calculate $\varphi^*\alpha$ we use the determinant formula

$$\det \varphi' = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \partial_{\sigma(1)} \varphi^1 \cdots \partial_{\sigma(n)} \varphi^n,$$

and the fact that for any $\sigma \in S_n$ we have

$$du^{\sigma(1)} \wedge \cdots \wedge du^{\sigma(n)} = \operatorname{sgn}(\sigma) du^1 \wedge \cdots \wedge du^n;$$

here S_n is the set of permutations of the set $\{1, \dots, n\}$, and $\operatorname{sgn}(\sigma)$ is 1 or -1 according as σ is an even or odd number of transpositions away from being the identity (see Section 1.4 of [Ta]).

$$\begin{aligned}
\varphi^*\alpha &= A(\varphi(u))d(\varphi^*x^1) \wedge \cdots \wedge d(\varphi^*x^n) \\
&= A(\varphi(u)) \sum_{k_1,\dots k_n=1}^{n} \partial_{k_1}\varphi^1(u)du^{k_1} \wedge \cdots \wedge \partial_{k_n}\varphi^n(u)du^{k_n} \\
&= A(\varphi(u)) \sum_{\sigma\in S_n} \partial_{\sigma(1)}\varphi^1(u)du^{\sigma(1)} \wedge \cdots \wedge \partial_{\sigma(n)}\varphi^n(u)du^{\sigma(n)} \\
&= A(\varphi(u)) \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\partial_{\sigma(1)}\varphi^1(u)du^1 \wedge \cdots \wedge \partial_{\sigma(n)}\varphi^n(u)du^n \\
&= A(\varphi(u)) \det(\varphi'(u))du^1 \wedge \cdots \wedge du^n,
\end{aligned}$$

and we say that φ is *orientation preserving* or *orientation reversing* according as $\det\varphi' > 0$ or $\det\varphi' < 0$.

3.4. **Integration of general forms.** Let $M \subset \mathbb{R}^n$ be a k-dimensional surface. We say that a k-form α_0 is *nowhere vanishing* on M if $\varphi^*\alpha_0$ is nowhere vanishing for any parametrization $\varphi\colon \mathcal{O} \to M$. If there exists a nowhere vanishing k-form on M, we say that M is *orientable*, and an *orientation* on M is a choice of such α_0 which is declared to be *positive*. If $\varphi\colon \mathcal{O} \to M$ is a parametrization, for any k-form α we put

$$\int_{\varphi(\mathcal{O})} \alpha = \int_{\mathcal{O}} \varphi^*\alpha,$$

with the orientation on $\mathcal{O}$ chosen such that $\int_{\varphi(\mathcal{O})} \alpha_0 > 0$.

For any surface M and any parametrization $\varphi\colon \mathcal{O} \to M$, we can define an orientation on $\varphi(\mathcal{O})$ by defining a k-form $\langle\alpha_0, (V_1, \dots, V_k)\rangle = \det((D\varphi)^T V)$, where V is the matrix whose columns are $V_1, \dots, V_k$; this form is nowhere vanishing on $\varphi(\mathcal{O})$ because we can take $V = D\varphi$. This orientation on $\varphi(\mathcal{O})$ corresponds to the standard orientation on $\mathcal{O}$, and we say that then $\varphi(\mathcal{O})$ is *oriented by* φ.

For example, orienting a curve $\gamma(I)$ by γ in this new sense, where $I = [a, b]$, is the same as orienting it from $\gamma(a)$ to $\gamma(b)$ in the familiar sense. Then $\gamma^*\alpha_0 = \det((D\gamma)^T D\gamma)dt = |D(\gamma)|^2 dt$

As another example, orient the sphere of radius R in $\mathbb{R}^3$ by the parametrization $\psi(\theta, \varphi) = (x(\theta, \varphi), y(\theta, \varphi), z(\theta, \varphi)) = (R\cos\theta\sin\varphi, R\sin\theta\sin\varphi, R\cos\varphi)$. Then

$$\psi^*dx = d(R\cos\theta\sin\varphi) = -R\sin\theta\sin\varphi d\theta + R\cos\theta\cos\varphi d\varphi,$$

$$\psi^*dy = d(R\sin\theta\sin\varphi) = R\cos\theta\sin\varphi d\theta + R\sin\theta\cos\varphi d\varphi.$$

$$\psi^*(dx \wedge dy) = -R^2\sin^2\theta\sin\varphi\cos\varphi d\theta \wedge d\varphi + R^2\cos^2\theta\sin\varphi\cos\varphi d\varphi \wedge d\theta = -\tfrac{1}{2}R^2\sin(2\varphi)d\theta \wedge d\varphi.$$

Hence if M is the upper hemisphere (the part where $\varphi < \pi/2$) then

$$\int_M dx \wedge dy = \int_0^{\pi/2}\int_0^{2\pi} -\frac{1}{2}R^2\sin(2\varphi)d\theta d\varphi = \frac{\pi}{2}R^2\cos(2\varphi)\Big|_0^{\pi/2} = -\frac{\pi}{2}R^2.$$

The classic example of a nonorientable surface is the Möbius band, which is parametrized by

$$\varphi(u, \theta) = a(\cos\theta, \sin\theta, 0) + b(u\sin(\theta/2)\cos\theta, u\sin(\theta/2)\sin\theta, u\cos(\theta/2)), \quad (u, \theta) \in [-1, 1] \times [0, 2\pi],$$

for some $a > b > 0$.

3.5. **Identifying forms with vector fields.** We identify one-forms with vector fields when we compute the work done by a force. For a curve C parametrized by $\gamma\colon (a,b) \to \mathbb{R}^n$, we write

$$\int_C \sum_{j=1}^n F_j dx^j = \int_a^b \sum_{j=1}^n F_j(\gamma(t)) \frac{\dot\gamma^j(t)}{|\dot\gamma(t)|} |\dot\gamma(t)| dt = \int_C F \cdot T ds. \tag{5}$$

Thus the integral of a one-form over an oriented curve C is the integral of the dot product of $F = (F_1, \dots, F_n)$ with the unit tangent vector T to the curve. The integrand is also called the projection of F in the direction T, and the integral is also called the circulation of F around C.

We identify $n-1$-forms with vector fields when we compute flux through a surface. This is done using the *Hodge star* operator, which is the linear map from k-forms to $(n-k)$-forms such that, for any permutation $\sigma \in S_n$, we have

$$*f(x) dx^{\sigma(1)} \wedge \cdots \wedge dx^{\sigma(k)} = \operatorname{sgn}(\sigma)\, f(x) dx^{\sigma(k+1)} \wedge \cdots \wedge dx^{\sigma(n)}.$$

Thus for one-forms on $\mathbb{R}^2$ we have

$$*(F_1 dx^1 + F_2 dx^2) = F_1 dx^2 - F_2 dx^1,$$

and for one- and two-forms on $\mathbb{R}^3$ we have

$$*(F_1 dx^1 + F_2 dx^2 + F_3 dx^3) = F_1 dx^2 \wedge dx^3 + F_2 dx^3 \wedge dx^1 + F_3 dx^1 \wedge dx^2,$$
$$*(F_1 dx^2 \wedge dx^3 + F_2 dx^3 \wedge dx^1 + F_3 dx^1 \wedge dx^2) = F_1 dx^1 + F_2 dx^2 + F_3 dx^3,$$

and in general, using the notation $dx^{j+n} = dx^j$,

$$* \sum_{j=1}^n F_j dx^j = \sum_{j=1}^n (-1)^{(j-1)(n-1)} F_j dx^{j+1} \wedge \cdots \wedge dx^{j+n-1}.$$

For an $n-1$-dimensional surface M parametrized by $\varphi\colon \mathcal{O} \to \mathbb{R}^n$, the analogue of (5) is

$$\begin{aligned}
\int_M * \sum_{j=1}^n F_j dx^j &= \int_M \sum_{j=1}^n (-1)^{(j-1)(n-1)} F_j(x) dx^{j+1} \wedge \cdots \wedge dx^{j+n-1} \\
&= \int_{\mathcal{O}} \sum_{j=1}^n (-1)^{(j-1)(n-1)} F_j(\varphi(u)) \det \left[\partial_k \varphi(u)^{j+\ell}\right]_{k,\ell=1}^{n-1} du^1 \wedge \cdots du^{n-1} \\
&= \int_{\mathcal{O}} \det \left[\begin{array}{c|c} F_1 & \\ \vdots & D\varphi(u) \\ F_n & \end{array} \right] du^1 \wedge \cdots du^{n-1} \\
&= \int_{\mathcal{O}} \left(\det \left[\begin{array}{c|c} F_1 & \\ \vdots & D\varphi(u) \\ F_n & \end{array} \right] \Big/ \sqrt{\det D\varphi(u)^T D\varphi(u)} \right) \sqrt{\det D\varphi(u)^T D\varphi(u)} \\
&= \int_M F_j \cdot N dS,
\end{aligned} \tag{6}$$

where N is a unit normal vector to M; this is because the columns of $D\varphi$ are all tangent vectors to M, and the numerator in the large parentheses is the the volume of the n-parallelipiped spanned by F and the columns of D_φ, and the denominator in the large parentheses is the volume of the $n-1$-parallelipiped spanned by the columns of $D\varphi$.

Thus the integral of the $n-1$-form $*(F_1 dx^1 + \cdots + F_n dx^n)$ over an oriented surface M is the integral of the dot product of $F = (F_1, ..., F_n)$ with the unit normal N to the surface. The integrand is also called the projection of F in the direction N, and the integral is also called the flux of F through M.

Let us write this out when $n = 2$ and $n = 3$. For $n = 2$, since $(-F_2, F_1)$ is the rotation of (F_1, F_2) by a right angle clockwise,

$$\int_M *(F_1 dx^1 + F_2 dx^2) = \int_{\mathcal{O}} \det \begin{bmatrix} F_1 & \frac{d\varphi^1}{du} \\ F_2 & \frac{d\varphi^2}{du} \end{bmatrix} du = \int_M (-F_2, F_1) \cdot T ds = \int_M F \cdot N ds,$$

where N is a unit normal vector, with the normal pointing a right angle counterclockwise from the unit tangent T. For $n = 3$, the integrand in the second line of (6) is

$$\sum_{j=1}^{3} F_j(\varphi(u)) \det \begin{bmatrix} \partial_1\varphi(u)^{j+1} & \partial_2\varphi(u)^{j+1} \\ \partial_1\varphi(u)^{j+2} & \partial_2\varphi(u)^{j+2} \end{bmatrix} du^1 \wedge du^2,$$

and the matrices in the fifth line are

$$\left[\begin{array}{c|c} F_1 & \\ \vdots & D\varphi(u) \\ F_n & \end{array}\right] = \begin{bmatrix} F_1(\varphi(u)) & \partial_1\varphi(u)^1 & \partial_2\varphi(u)^1 \\ F_2(\varphi(u)) & \partial_1\varphi(u)^2 & \partial_2\varphi(u)^2 \\ F_3(\varphi(u)) & \partial_1\varphi(u)^3 & \partial_2\varphi(u)^3 \end{bmatrix},$$

and

$$[D\varphi(u)^T D\varphi(u)] = \begin{bmatrix} \partial_1\varphi(u) \cdot \partial_1\varphi(u) & \partial_1\varphi(u) \cdot \partial_2\varphi(u) \\ \partial_1\varphi(u) \cdot \partial_2\varphi(u) & \partial_2\varphi(u) \cdot \partial_2\varphi(u) \end{bmatrix}.$$

The direction of the normal N is determined from $\partial_1\varphi$ and $\partial_2\varphi$ by the right hand rule, i.e. it is the direction of the cross product $\partial_1\varphi \times \partial_2\varphi$.

As an example, consider the flux of the vector field (x^3, y^2, z) through the part of the graph of $f(x, y, z) = x^2 - y^2$ where $|x| < 1$ and $|y| < 1$, oriented upward. We have

$$\int_M *(x^3 dx + y^2 dy + z dz) = \int_M x^3 dy \wedge dz + y^2 dz \wedge dx + z dx \wedge dy. \tag{7}$$

We use the parametrization $\varphi(u, v) = (u, v, u^2 - v^2)$, so that

$$\varphi^* dx = du, \qquad \varphi^* dy = dv, \qquad \varphi^* dz = 2u du - 2v dv,$$

and we put $\mathcal{O} = \{(u, v) \colon |u| < 1 \text{ and } |v| < 1\}$. Then (7) becomes

$$\int_{\mathcal{O}} u^3 dv \wedge (2u du - 2v dv) + v^2(2u du - 2v dv) \wedge du + (u^2 - v^2) du \wedge dv,$$

which, using $du \wedge du = dv \wedge dv = 0$ and $du \wedge dv = -dv \wedge du$, becomes

$$\int_{\mathcal{O}} (-2u^4 + 2v^3 + u^2 - v^2) du \wedge dv = \pm \int_{-1}^{1} \int_{-1}^{1} (-2u^4 + 2v^3 + u^2 - v^2) du dv.$$

To decide between plus and minus, we use the fact that upward orientation means that the flux of $(0, 0, 1)$ should be positive, i.e.

$$\int_M *dz = \int_M dx \wedge dy = \int_{\mathcal{O}} du \wedge dv = \pm \int_{-1}^{1} \int_{-1}^{1} du dv > 0,$$

so we need the plus sign, and the answer is

$$\int_{-1}^{1} \int_{-1}^{1} (-2u^4 + 2v^3 + u^2 - v^2) du dv = \int_{-1}^{1} \int_{-1}^{1} -2u^4 du dv = -4/5.$$

3.6. **Stokes' Theorem.** Stokes' Theorem, proved in Section 4.3 of [Ta], says that

$$\int_{\partial M} \alpha = \int_M d\alpha, \tag{8}$$

where M is a compact, oriented k-dimensional surface, and α is a $k-1$-form. In Section 4.4 of [Ta], the classical Green, Stokes, and Gauss formulas are derived from it.

For example, if $\alpha = A_1 dx^1 + A_2 dx^2$ and M is a bounded open set in $\mathbb{R}^2$ with smooth boundary, then $d\alpha = (\partial_1 A_2 - \partial_2 A_1) dx^1 \wedge dx^2$ and Stokes' Theorem (8) reduces to Green's Theorem

$$\int_{\partial M} A_1 dx^1 + A_2 dx^2 = \int_M (\partial_1 A_2 - \partial_2 A_1) dx^1 \wedge dx^2.$$

Another important special case of (8) is the case where M is a bounded open set in $\mathbb{R}^n$ with smooth boundary, and $\alpha = * \sum_{j=1}^n F_j dx^j$ as in (6). Then we get

$$\begin{aligned} d\alpha &= \sum_{j=1}^n (-1)^{(j-1)(n-1)} dF_j \wedge dx^{j+1} \wedge \cdots \wedge dx^{j+n-1} \\ &= \sum_{j=1}^n (-1)^{(j-1)(n-1)} (\partial_j F_j) dx^j \wedge dx^{j+1} \wedge \cdots \wedge dx^{j+n-1} \\ &= (\partial_1 F_1 + \cdots + \partial_n F_n) dx^1 \wedge \cdots dx^n \end{aligned}$$

Combining with (6) and plugging into (8) gives

$$\int_{\partial M} F \cdot N dS = \int_M (\partial_1 F_1 + \cdots + \partial_n F_n), \tag{9}$$

for the right choice of unit normal vector field N to ∂M. We can see that this choice is the one where N points away from M (the *outward pointing normal*) by looking at the basic case where $M = \{(x^1, \ldots x^n) \colon x^1 < 0\}$, as in (4.3.1) of [Ta]. Then, if $F = (x^1, 0, \ldots, 0)$, the integrands on both sides of (9) are positive with this choice of normal.

3.7. **Volumes of balls and spheres.** We denote the open ball centered at p with radius R by

$$B_p(R) = \{x \in \mathbb{R}^n \colon |x - p| < R\}.$$

Then, applying (9) with $M = B_0(R)$ and $F = (x^1, \ldots, x^n)$ gives

$$\int_{\partial B_0(R)} R dS = \int_{B_0(R)} n,$$

because $N = F/R$ and $|F| = R$ on $\partial B_0(R)$. Thus, letting $V_n(R)$ be the volume of $B_0(R)$ and $A_{n-1}(R)$ be the surface area of $\partial B_0(R)$, we get

$$R A_{n-1}(R) = n V_n(R).$$

From (3.2.32) of [Ta] we know that

$$A_{n-1}(1) = 2\pi^{n/2} / \Gamma(n/2).$$

And, since $V_n(1) = A_{n-1}(1)/n$, that gives a formula for $V_n(1)$ as well. We can get $V_n(R)$ by changing variables

$$V_n(R) = \int_{B_0(R)} 1 = \int_{B_0(1)} |\det g'| = \int_{B_0(1)} R^n = R^n V_n(1),$$

using $g(x) = Rx$. Rewriting this in terms of A_{n-1} gives

$$A_{n-1}(R) = R^{n-1}A_{n-1}(1).$$

3.8. **Harmonic functions.** If $F = \nabla u$ for some function u, then (9) becomes

$$\int_{\partial M} \nabla u \cdot N dS = \int_M \Delta u,$$

where $\Delta u = (\partial_1^2 + \cdots \partial_n^2)u$ is the Laplacian of u. If $\Delta u = 0$ in some open set Ω we say that u is *harmonic* and harmonic functions obey the following *mean value property*:

$$u(p) = \frac{1}{A_{n-1}(1)} \int_{\partial B_0(1)} u(p + r\omega) dS(\omega), \tag{10}$$

for any $p \in \Omega$ and for any $r > 0$ such that $\overline{B_p(r)} \subset \Omega$. To prove it, differentiate the right hand side with respect to r to get

$$\begin{aligned} \frac{1}{A_{n-1}(1)} \int_{\partial B_0(1)} \partial_r u(p + r\omega) dS(\omega) &= \frac{1}{A_{n-1}(1)} \int_{\partial B_0(1)} \nabla u(p + r\omega) \cdot \omega dS(\omega) \\ &= \frac{1}{A_{n-1}(1)} \int_{\partial B_0(1)} \nabla u \cdot N dS = 0. \end{aligned}$$

Hence the right hand side of (10) is independent of r, and it simplifies to equals the left hand side when $r = 0$, hence the two sides are equal for all $r \geq 0$, as long as we stay in the set where $\Delta u = 0$. (See also Proposition 5.1.5 of [Ta].)

References

[Ta] Michael E. Taylor, *Introduction to Analysis in Several Variables*, AMS Sally Series of Pure and Applied Undergraduate Texts 46, 2020. Preprint available online at `https://mtaylor.web.unc.edu/wp-content/uploads/sites/16915/2018/04/analmv.pdf`.