MA 442 final review problems

Finalized version as of April 27th

The final exam will be in STON 217 from 7 pm to 9 pm on Friday, May 7th. No notes or electronic devices are allowed. The problems on the exam will be closely based on parts of problems from the list below. For each problem, you must justify your answers. Please let me know if you have a question or find a mistake.

1. Let A be an $m \times n$ matrix, and let $f \colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ be given by $f(x, y) = x \cdot (Ay)$. Given (a, b) in $\mathbb{R}^m \times \mathbb{R}^n$ and (x, y) in $\mathbb{R}^m \times \mathbb{R}^n$, find $Df(a, b)(x, y)$

2. Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be nonempty open sets. State and use a fact from linear algebra to show that if $f \colon U \to V$ is invertible, with f and f^{-1} both differentiable, then $m = n$.

3. Consider the system of equations

$$\begin{aligned} xyz &= \alpha \\ x + y + z &= \beta \\ x^{y^z} &= \gamma. \end{aligned}$$

   Given $a > 2$, find the solution of the system when $(\alpha, \beta, \gamma) = (0, a, 2)$. Show that, for (α, β, γ) sufficiently near $(0, a, 2)$, the system defines (x, y, z) uniquely as a function of (α, β, γ). (Take advantage of the partial derivatives that are zero to simplify the calculation.) Find $\partial_\alpha z$, $\partial_\beta z$, $\partial_\gamma z$, $\partial_\beta x$, $\partial_\gamma x$, $\partial_\beta y$, and $\partial_\gamma y$ at $(0, a, 2)$. (Note that these are easier than $\partial_\alpha x$ and $\partial_\alpha y$.)

4. Let $a > 0$ be given, and let M be the torus in $\mathbb{R}^4$ given by the equations

$$w^2 + x^2 = y^2 + z^2 = a^2.$$

   Find the surface area of the torus. Find the lengths of the six shortest geodesics on M from $(w, x, y, z) = (a, 0, a, 0)$ to $(w, x, y, z) = (a, 0, 0, a)$. Show that, for any real number N, there are non-self-intersecting geodesics on M from $(w, x, y, z) = (a, 0, a, 0)$ to $(w, x, y, z) = (a, 0, 0, a)$ of length greater than N and find the length of one such geodesic.

5. Let $a > b > 0$ be given. Find $|\int_C \alpha|$, where $\alpha = xdy - ydx$, and C is the boundary edge of the Möbius band given by

$$\varphi(u, \theta) = a(\cos\theta, \sin\theta, 0) + b(u \sin(\theta/2) \cos\theta, u \sin(\theta/2) \sin\theta, u \cos(\theta/2)),$$

   where $(u, \theta) \in [-1, 1] \times [0, 2\pi]$.
   Hint: Use $x \frac{d}{d\theta} y - y \frac{d}{d\theta} x = x^2 \frac{d}{d\theta}(y/x)$ and $\int_0^{2\pi} \sin^2 t dt = \pi$.

6. Let $\varphi(u, v) = (x(u, v), y(u, v), z(u, v)) = (u^2 - v^2, u^2 + v^2, uv)$. For which real numbers a and b is $\varphi^*[a(x + y)dx \wedge dy + bz(dx + dy) \wedge dz] = 0$? (For these values the differential form in brackets corresponds to a vector field tangent to the surface parametrized by φ.)

7. Let M be a sphere in $\mathbb{R}^3$ which encloses a ball of volume 5. Let M be oriented so that $\int_M xdy \wedge dz > 0$. Use Stokes' theorem to evaluate

$$\int_M e^z dy \wedge dz + (\cos x + 2y) dz \wedge dx + (3x^2 - 4y^2) dx \wedge dy.$$

8. Let $a > 0$ and $b > 0$ be given. Let C_1 be the curve in $\mathbb{R}^4$ given by

$$w^2 + x^2 = a^2, \qquad y^2 + z^2 = b^2, \qquad bx = ay, \qquad wz \geq 0,$$

and C_2 be the curve in $\mathbb{R}^4$ given by

$$w^2 + x^2 = a^2, \qquad y^2 + z^2 = b^2, \qquad bx = ay, \qquad wz \leq 0.$$

Find the lengths of C_1 and C_2. Find

$$\int_{C_j} z dx - y dw - y dz + w dx + z dy - x dw,$$

for $j = 1$ and $j = 2$, in each case with the curve oriented so that x is increasing when $w = a$ and $x = y = 0$.

9. Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function. Write out and simplify df, $*df$, ddf, $d{*df}$, $df \wedge df$, and $df \wedge {*df}$ in terms of the partial derivatives of f. If one of these forms is identically zero, which of the others must also be? (Recall that $*(F_1 dx^1 + F_2 dx^2) = -F_2 dx^1 + F_1 dx^2$.)