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MA 442
Spring 2021

Homework 1

Due January 27th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake.

1. Let K_0 be the segment from $(0,0)$ to $(1,0)$ in $\mathbb{R}^1$. Let K_1 be the four equal segments from $(0,0)$ to $(1/3,0)$ to $(1/2,\sqrt{1/12})$ to $(2/3,0)$ to $(1,0)$. More generally, for each $n \ge 1$, construct K_n from K_{n-1} by the following operation. For each segment S of K_{n-1}, let M be the middle third of S. Then replace M with two segments M' and M'' having the same length as M and such that M, M', and M'' form an equilateral triangle. See the next page for pictures of $K_1, \ \ldots, K_5$.

   Let L_n be the length of K_n.

   (a) Find L_n for each n.

   (b) Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be a continuous function such that,[1] if p and q are the endpoints of a segment of K_m for some m, then there are s and t such that $0 \le s < t \le 1$, $\gamma(s) = p$, and $\gamma(t) = q$. Show that the length of γ is infinite.

2. Exercise 1.1 of `https://www.math.purdue.edu/~kdatchev/442/geodesics.pdf`.

3. Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let (a_{ij}) be the entries of the corresponding matrix. The *norm* of A is given by

$$\|A\| = \sup_{x \in \mathbb{R}^n \setminus 0} \frac{|Ax|}{|x|} = \sup_{|x|=1} |Ax| = \sup_{|x| \le 1} |Ax|.$$

   (a) Show that all three suprema are equal.

   (b) Find an explicit upper bound for $\|A\|$ in terms of the (a_{ij}).

   (c) Prove that A is continuous by using $\|A\|$ to verify the ε-δ condition.

4. Let

$$f(x,y) = \begin{cases} 1, & 0 < |y| < x^2, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Show that f is not continuous at 0.

   (b) Show that $\lim_{t \to 0} f(at, bt) = 0$ for any real a and b by finding an explicit interval I in terms of a and b, as large as possible, such that $f(at, bt) = 0$ when $t \in I$.

[1] Such a function was constructed by Koch in his paper `https://www.math.purdue.edu/~kdatchev/442/koch.pdf`, which also has further interesting geometric constructions.