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MA 442
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## Homework 3

Due February 10th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake.

1. Given $b \in \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}$ continuously differentiable, consider the *transport equation*:
$$\partial_t u(t,x) + b \cdot \partial_x u(t,x) = 0, \qquad u(0,x) = f(x),$$
where $\partial_x u = (\partial_{x_1} u, \dots, \partial_{x_n} u)$.
   (a) In the case $n = 1$ and $b = 2$, along which lines in the $(t,x)$ plane must any solution $u$ be constant? Sketch a picture.
   (b) Returning to the general case, find a linear $\varphi\colon \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that any solution $u$ must obey $u(t,x) = u(0, \varphi(t,x))$ for all $t$ and $x$; use the mean value theorem to justify your answer. Give an explicit formula for the unique solution $u$ in terms of $f$ and check that it is a solution.
2. Let $a > 0$ be given, and consider the equation $xe^x = a$.
   (a) Show that $xe^x = a$ has a unique real solution $x^*$. Show that $x^*$ is positive and the sign of $x^* - 1$ is the same as the sign of $a - e$.
   (b) Show that $x^*$ is the unique fixed point of the map $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = ae^{-x}$. Use the mean value theorem to show that $x^*$ lies between $x$ and $f(x)$ for any $x$.
   (c) Show that $f\colon I \to I$ when $I = [0, \infty)$. For which $a$ is $|f'| < 1$ on $I$? What is the maximum of $|f'|$ on $I$?
   (d) The map $f$ was obtained by isolating one of the $x$'s in the equation $xe^x = a$. Define $g(x)$ by isolating the other $x$. What are the domain and range of $g$? Use the mean value theorem to show that $x^*$ lies between $x$ and $g(x)$ for any $x$.
   (e) Show that if $a > e$, then $g\colon I \to I$ and $|g'| < 1$ on $I$, where $I = [\ln a - \ln\ln a, \ln a]$. What is the maximum of $|g'|$ on $I$?
   (f) Let $a = 1/2$ and let $x_0 = 1$. Use a calculator write down decimal approximations of the initial terms of the sequence $x_k = f(x_{k-1})$. Include enough digits and enough terms to determine the first two digits of $x^*$.
   (g) Repeat part (f), but with $a = 100$, $x_0 = \ln 100$, and $x_k = g(x_{k-1})$.