

### Homework 4

Due February 8th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake. There are some hints on the second page.

1. Let  $X$  be a metric space, let  $\lambda \in [0, 1)$ , and let  $f: X \rightarrow X$  obey  $d(f(x), f(y)) \leq \lambda d(x, y)$  for all  $x$  and  $y$  in  $X$ . Prove that  $f$  is continuous and that there is at most one  $x$  in  $X$  such that  $f(x) = x$ .
2. Let  $X$  be a compact metric space and let  $f: X \rightarrow X$  obey  $d(f(x), f(y)) < d(x, y)$  for all  $x$  and  $y$  in  $X$  such that  $x \neq y$ . Prove that there is a unique  $x \in X$  such that  $f(x) = x$ .
3. Let  $X = [0, 1]$  with  $d(x, y) = |x - y|$  and  $f: X \rightarrow X$  be given by  $f(x) = x - x^2$ . Prove that the assumptions of Problem 2 are satisfied but not the assumptions of Problem 1.
4. Shifrin 6.2.6, 6.2.7a, 6.2.8, 6.2.11

*Hints:*

2. Prove that the function  $x \mapsto d(x, f(x))$  is continuous and has a minimum value of zero.
3. Observe that  $d(f(x), f(y))/d(x, y)$  simplifies nicely.
- 6.2.6. Use the implicit function theorem to compute the partial derivative of the angle with respect to each of the sides in turn, and show that one of those three partial derivatives is bigger than the other two regardless of the triangle.
- 6.2.8 Differentiate  $F(p, V, T(p, V)) = 0$  with respect to  $p$ ,  $F(p, V(p, T), T) = 0$  with respect to  $T$ , etc., and simplify the resulting system of equations.
- 6.2.11 Mimic the argument in the first paragraph of the proof of Theorem 2.1.