Kiril Datchev
MA 442
Spring 2021

Homework 4

Due February 24th on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake.

1. Let $0 < r < R$, and let C_r and C_R be the circles in $\mathbb{R}^2$ centered at the origin with radii r and R. Let S be the set of points in $\mathbb{R}^2$ which are midpoints of segments having one endpoint on C_r and the other on C_R.
   (a) Denote points on the circles C_r and C_R by $(r\cos\theta, r\sin\theta)$ and $(R\cos\varphi, R\sin\varphi)$, where θ and φ are real parameters. Give formulas $x = x(\theta,\varphi)$ and $y = y(\theta,\varphi)$ for the corresponding points of S.
   (b) The inverse function theorem implies that the equations for x and y can be solved for θ and φ, as long as the derivative of the map $(\theta,\varphi) \to (x,y)$ is invertible. Find the exceptional values of (θ,φ) where this invertibility fails.
   (c) Find two circles C and C', consisting of precisely the points of S corresponding to the exceptional values of (θ,φ) from part (b).
   (d) Find an explicit point of S in the region strictly between C and C', and then find a point inside both circles and a point outside both circles which cannot lie in S.
   (e) You don't have to write anything for this part, but the above results can be used to show that S consists precisely of the points on or strictly between the circles C and C'. This is because, by part (b), for any $\gamma\colon [0,1] \to \mathbb{R}^2$ with $\gamma(0) \in S$, if we put $t_\gamma = \sup\{t \in [0,1] \mid \gamma(s) \in S \text{ for all } s \in [0,t]\}$, then $t_\gamma < 1$ implies $\gamma(t_\gamma) \in C \cup C'$.
2. The inverse functon theorem says that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $f'(a)$ invertible for some $a \in \mathbb{R}^n$, then there are $\delta > 0$ and a C^1 function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $g(f(x)) = x$ and $f(g(y)) = y$ when $|x-a| < \delta$ and $|y - f(a)| < \delta$.
   (a) Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Show that if $F'(a)$ has a left inverse, i.e. there is a linear $L\colon \mathbb{R}^m \to \mathbb{R}^n$ such that $LF'(a) = I$, then there are $\delta > 0$ and a C^1 function G such that $G(F(x)) = x$ for when $|x-a| < \delta$.
   (b) Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be C^1. Show that if $F'(a)$ has a right inverse with a in its range, i.e. there is a linear $R\colon \mathbb{R}^m \to \mathbb{R}^n$ such that $F'(a)R = I$ and $Rb = a$ for some $b \in \mathbb{R}^m$, then there are $\delta > 0$ and a C^1 function G such that $F(G(y)) = y$ for when $|y - F(a)| < \delta$.
3. Let $\xi \in \mathbb{R}^n$ and $a \in \mathbb{R}$ be given. Consider the initial value problem $\dot{x}(t) = ax(t)$, $x(0) = \xi$.
   (a) Define $x_0(t) \equiv \xi$, and, for $k \geq 0$, $x_{k+1}(t) = \xi + \int_0^t a x_k(s)ds$. Find x_1 and x_2, and

give a general formula for x_k.

(b) Evaluate $x(t) = \lim_{k\to\infty} x_k(t)$ and verify that $\dot{x}(t) = ax(t)$, $x(0) = \xi$.

(c) Repeat parts (a) and (b), but with $x_0(t) \equiv \xi$ replaced by $x_0(t) \equiv \eta$, where η is some other point in $\mathbb{R}^n$.