Kiril Datchev
MA 442
Spring 2021

Homework 5

Due March 3rd on paper at the beginning of class. Justify your answers. Please let me know if you have a question or find a mistake.

1. Prove the statement in the footnote on page 3 of `https://www.math.purdue.edu/~kdatchev/442/ode.pdf`, i.e. show that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f(x) > 0$ for some $x > 0$, then there is $y \in (0, x)$ such that $f(y) > 0$ and $f'(y) > 0$.
   Hint: Apply the mean value theorem to f on $[z, x]$, where z is the greatest zero of f on $[0, x]$.
2. Exercise 2 from page 4 of `https://www.math.purdue.edu/~kdatchev/442/ode.pdf`.
3. Exercise 3 from page 4 of `https://www.math.purdue.edu/~kdatchev/442/ode.pdf`.
4. Let a, b, c, d, be given positive constants, and let f, g be given C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Consider the differential equations

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1x_2 + \varepsilon f(x_1, x_2), \qquad & x_1(0, \varepsilon) = \xi_1, \\ \dot{x}_2 &= -cx_2 + dx_1x_2 + \varepsilon g(x_1, x_2), \qquad & x_2(0, \varepsilon) = \xi_2, \end{aligned}$$

   where ε is a small parameter, and $\xi_1 \geq 0$ and $\xi_2 \geq 0$ are initial conditions. Here x_1 is a population of prey, and x_2 a population of predators.

   (a) Find $x_1(t, 0)$ and $x_2(t, 0)$ when $\xi_2 = 0$. (Prey thrives without predators.)
   Hint: Guess and check (based on the last problem of the previous homework) that $x_2(t, 0) = 0$ and $x_1(t, 0) = \xi_1 e^{at}$ works for all t; the solution is unique by the existence and uniqueness theorem.

   (b) Use the same approach to find $x_1(t, 0)$ and $x_2(t, 0)$ when $\xi_1 = 0$. (Predators die out without prey.)

   (c) Find the unique $\xi_1^* > 0$ and $\xi_2^* > 0$ such that $x_j(t, 0) = \xi_j^*$ for $j = 1$ and 2, and for all t. (This is the coexistence equilibrium.)

   (d) Use the implicit function theorem to show that there exist $\varepsilon_0 > 0$ and C^1 functions $\varphi_1\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $\varphi_2\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ such that $\varphi_j(0) = \xi_j^*$ and $x_j(t, \varepsilon) = \varphi_j(\varepsilon)$ for $j = 1$ and 2, and for all t. (Adding a small correction perturbs the equilibrium slightly.)

   (e) Find $\varphi_1'(0)$ and $\varphi_2'(0)$ in terms of a, b, c, d, f, g. (This gives the first order change in the equilibrium).