THE INVERSE AND IMPLICIT FUNCTION THEOREMS

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Implicit Function Theorem. *Let* $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m$ *be continuously differentiable at* (a, b)*, with* $\partial_x F(a, b)$ *invertible. Then there is a continuously differentiable* $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ *such that* $\varphi(b) = a$ *and* $F(\varphi(y), y) = F(a, b)$ *for* y *near* b*.*

Inverse Function Theorem. *Let* $f\colon \mathbb{R}^n \to \mathbb{R}^n$ *be continuously differentiable at* a*, with* $f'(a)$ *invertible. Then there is a continuously differentiable* $g\colon \mathbb{R}^n \to \mathbb{R}^n$ *such that* $g(f(x)) = x$ *for* x *near* a *and* $f(g(y)) = y$ *for* y *near* $f(a)$*.*

To prove the second theorem from the first, take $F(x, y) = (x, f(x))$. To prove the first theorem from the second, take $f(x, y) = (F(x, y), y)$.

Example. Consider a chain of four linked rods, having lengths a, b, c, d, and with the ends of the chain fixed at $(0, 0$ and $(R, 0)$, with $R < a + b + c + d$. Let α, β, γ, δ be the angles with the horizontal made by the four rods, as in the picture below.

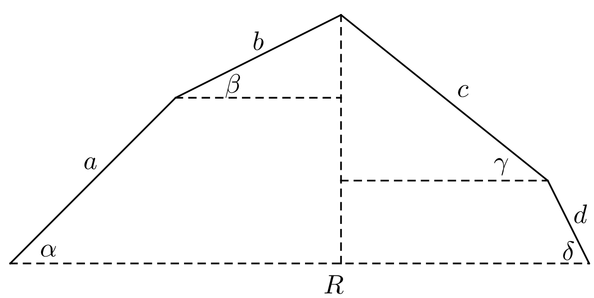

Figure 1. The four linked rods.

We will find the region which can be reached by the middle vertex. Computing the width and height of the figure in two ways each, we obtain a system of equations relating the angles

$$a \cos\alpha + b \cos\beta + c \cos\gamma + d \cos\delta = R,$$
$$a \sin\alpha + b \sin\beta - c \sin\gamma - d \sin\delta = 0.$$

Denote by $F_1 = F_1(\alpha, \beta, \gamma, \delta)$ the left hand side of the first equation, and by $F_2 = F_2(\alpha, \beta, \gamma, \delta)$ the left hand side of the second equation, so that $F = (F_1, F_2)\colon \mathbb{R}^4 \to \mathbb{R}^2$. Applying the implicit function theorem to F, we see that we can write $\varphi(\alpha, \beta) = (\gamma, \delta)$ near any point (α_0, β_0) as long as $\partial_\gamma F_1 \partial_\delta F_2 - \partial_\gamma F_2 \partial_\delta F_1 \neq 0$, i.e. as long as $\gamma - \delta$ is not a multiple of π. Suppose for simplicity that $\gamma = \delta$ is the only possibility. Thus if $(\alpha, \beta) = (\alpha_0, \beta_0)$ is a possible configuration of the rods, then so is any (α, β) sufficiently near (α_0, β_0), unless the corresponding (γ_0, δ_0) obeys $\gamma_0 = \delta_0$.

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To see what region this corresponds to in terms of the middle vertex, write the coordinates of this vertex as

$$x = a\cos\alpha + b\cos\beta,$$
$$y = a\sin\alpha + b\sin\beta.$$

Denote by $f_1 = f_1(\alpha, \beta)$ the right hand side of the first equation, and by $f_2 = f_2(\alpha, \beta)$ the right hand side of the second equation, so that $f = (f_1, f_2)\colon \mathbb{R}^2 \to \mathbb{R}^2$. Applying the inverse function theorem to f, we see that we can write $g(x, y) = (\alpha, \beta)$ near any point (x_0, y_0) as long as $\partial_\alpha f_1 \partial_\beta f_2 - \partial_\alpha f_2 \partial_\beta f_1 \neq 0$, i.e. as long as $\alpha - \beta$ is not a multiple of π. Suppose again for simplicity that the only possibility is $\alpha = \beta$. Thus if (x_0, y_0) is a possible configuration of the rods, then so is any (x, y) sufficiently near (x_0, y_0) unless the corresponding $(\alpha_0, \beta_0, \gamma_0, \delta_0)$ obeys $\alpha_0 = \beta_0$ or $\gamma_0 = \delta_0$.

Thus to find the boundary of the region that can be reached by the middle vertex, we must find for which (x, y) we have $\alpha = \beta$ and for which we have $\gamma = \delta$.

If $\alpha = \beta$ then $x = (a + b)\cos\alpha$ and $y = (a + b)\sin\alpha$, i.e. x and y belong to the circle centered at 0 of radius $a + b$. Denote this circle C_1. If $\gamma = \delta$ then $x = R - (c + d)\cos\gamma$ and $y = (c + d)\sin\gamma$, i.e. x and y belong to the circle centered at $(R, 0)$ of radius $c + d$. Denote this circle C_2.

Thus the boundary of the region that can be reached by the middle vertex is a subset of $C_1 \cup C_2$. Since $(R/2, 0)$ is clearly reachable, the region inside both circles, i,e, the region defined by

$$x^2 + y^2 \leq (a + b)^2, \qquad (x - R)^2 + y^2 \leq (c + d)^2,$$

is reachable. Points outside that region are not reachable, since the triangle inequality shows that the middle vertex must be distance $\leq a + b$ from the origin and $\leq c + d$ from $(R, 0)$.

Contraction mappings. Let $K \subset \mathbb{R}^n$ be closed, and let $F\colon K \to K$, be a *contraction*, meaning there is $\lambda < 1$ such that

$$|F(x) - F(\tilde{x})| \leq \lambda |x - \tilde{x}|, \qquad \text{for all } x,\ \tilde{x} \in K. \tag{1}$$

A *fixed point* is a solution to $x = F(x)$.

Theorem. *Let $F\colon K \to K$ be a contraction. Then F has a unique fixed point, and any sequence $x_k = F(x_{k-1})$ converges to it, regardless of the initial point $x_0 \in K$*

Proof. By (1), if x and $\tilde{x}$ are both fixed points, then $|x - \tilde{x}| = 0$. To show existence, it is enough to show that the sequence converges; we conclude the limit x^* is a fixed point by taking the limit of the equation $x_k = F(x_{k-1})$. To show convergence, we write, for $k > 1$,

$$|x_k - x_{k-1}| = |F(x_{k-1}) - F(x_{k-2})| \leq \lambda |x_{k-1} - x_{k-2}| \leq \cdots \leq \lambda^{k-1} |x_1 - x_0|,$$

and so, if $m > k > 1$, then

$$|x_m - x_k| \leq \sum_{j=k+1}^{m} |x_j - x_{j-1}| \leq |x_1 - x_0| \sum_{j=k+1}^{m} \lambda^{j-1}.$$

Since this goes to 0 as $k \to \infty$, it follows that the sequence is Cauchy and hence convergent. $\square$

A sufficient condition for (1) is that F be continuously differentiable on K with $\|F'\| \le \lambda$. Indeed, then

$$\begin{aligned} |F(x) - F(\tilde{x})| &= \Big| \int_0^1 \frac{d}{dt} F(tx + (1-t)\tilde{x}) dt \Big| = \Big| \int_0^1 F'(tx + (1 - t\tilde{x})(x - \tilde{x}) dt \Big|, \\ &\le \Big| \int_0^1 F'(tx + (1 - t\tilde{x})(x - \tilde{x}) dt \Big| \le \int_0^1 |F'(tx + (1 - t\tilde{x})(x - \tilde{x})| \qquad (2) \\ &\le \int_0^1 \|F'(tx + (1 - t\tilde{x})\| |x - \tilde{x}| dt \le \int_0^1 \lambda |x - \tilde{x}| dt \le \lambda |x - \tilde{x}|. \end{aligned}$$

Example. Consider the equation $\frac{e^x}{x^2+2} = a$, where a is given. We wish to solve for x. To be able to apply the contraction mapping theorem, we rewrite this as a perturbation of the identity. More specifically, clear the denominator and take the logarithm of both sides to see that x is a solution if and only if it is a fixed point of $F(x) = \ln a + \ln(x^2 + 2)$. Since

$$|F'(x)| = \frac{2|x|}{2 + x^2} \le \frac{1 + x^2}{2 + x^2} \le \frac{1}{2},$$

where for the first inequality we used $(1 - |x|)^2 \ge 0$, we see that F has a unique fixed point and any sequence $x_k = F(x_{k-1})$ will converge to it. For example, with $a = 3$, $x_0 = 10^{100}$, we get, to two digits,

$$x_1 \approx 230,\ x_2 \approx 13,\ x_3 \approx 7.5,\ x_4 \approx 6.4,\ x_5 \approx 6.1,\ x_6 \approx 6.0,\ x_7 \approx 5.9,\ x_8 \approx 5.9,\ x_8 \approx 5.9,\ \dots$$

Proof of Inverse Function Theorem. We give the proof in the special case $a = 0$, $f'(a) = I$, and then deduce the general case from it. Below, $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$.

Take $\delta > 0$ such that

$$|x| \le 2\delta \implies \|f'(x) - I\| \le 1/2.$$

Then, when $|y| \le \delta$, apply the contraction mapping principle to the sequence

$$x_k = F(x_{k-1}) = x_{k-1} + y - f(x_{k-1}), \qquad x_0 = 0,$$

with $K = \{x \colon |x| \le 2\delta\}$. Since $\|F'\| \le 1/2$ on K, we have, by (2),

$$|f(x) - f(\tilde{x}) - (x - \tilde{x})| \le |x - \tilde{x}|/2, \qquad \text{for all } x,\ \tilde{x} \in K. \qquad (3)$$

We now solve $f(x) = y$, when $|y| \le \delta$, by defining the sequence

$$x_0 = 0, \qquad x_k = x_{k-1} + y - f(x_{k-1}),$$

and taking the limit as $k \to \infty$. To see that this limit exists, note that

$$|x_1| = |y| \le \delta.$$

Moreover, if $k > 1$ and $|x_j| \le 2\delta$ when $j < k$ then, by (3),

$$|x_k - x_{k-1}| = |x_{k-1} - f(x_{k-1}) - (x_{k-2} - f(x_{k-2}))| \le |x_{k-1} - x_{k-2}|/2 \le \delta 2^{1-k}.$$

Hence

$$|x_k| \le \sum_{j=1}^{k} |x_j - x_{j-1}| \le \delta \sum_{j=1}^{k} 2^{1-j} \le 2\delta.$$

Thus x_k converges to a point x with $|x| \le 2\delta$, and $f(x) = y$.

It remains to show that the inverse $g(y) = x$, now defined when $|y| \le \delta$, is continuously differentiable when $|y| < \delta$. When $|y| < \delta$ and $|k| < \delta - |y|$, let $x = g(y)$ and $h = g(y + k) - x$, so that

$$g(y) = x, \quad g(y + k) = x + h, \quad \text{so that} \quad f(x) = y, \quad f(x + h) = y + k.$$

Then, because f is differentiable at x,

$$k = f(x+h) - f(x) = f'(x)h + r, \quad \text{where} \quad |r|/|h| \to 0 \text{ as } |h| \to 0.$$

By (3) with $\tilde{x}$ replaced by $x+h$, we have $|k-h| \le |h|/2$, and hence $|h|/2 \le |k| \le 3|h|/2$. Thus

$$g(y+k) = x + h = x + f'(x)^{-1}k - f'(x)^{-1}r,$$

and $|f'(x)^{-1}r|/|k| \le 2|r|/|k| \to 0$ as $|k| \to 0$. That shows that $g'(y) = f'(g(y))^{-1}$, which is a continuous function of y. □

Exercise. Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be C^1, let $a \in \mathbb{R}^m$ and $b \in \mathbb{R}^n$ be such that $f(a) = b$, and suppose $f'(a)$ has full rank.

(1) If $n \ge m$, show that there are an open neighborhood U of a and $g\colon f(U) \to \mathbb{R}^n$ such that $g(f(x)) = x$ for all $x \in U$ and g is C^1 with $g'(y)f'(g(y)) = I$.
Hint: Apply the Theorem with f replaced by $T \circ f$ or $f \circ T$, where T is a suitable affine function.

(2) If $n \le m$, show that there are an open neighborhood V of b and $g\colon V \to \mathbb{R}^m$ such that $f(g(y)) = y$ for all $y \in V$ and g is C^1 with $f'(g(y))g'(y) = I$.

(3) If $n = m$, show that there are open neighborhoods U of a, V of b, and a function $g\colon V \to \mathbb{R}^n$ such that $g(f(x) = x$ for all $x \in U$, $f(g(y)) = y$ for all $y \in V$, and g is C^1 with $f'(g(y))g'(y) = I$.

The example is a variant of Examples 2.10.6 and 3.1.8 of [HuHu], and the proof of inverse function theorem follows Theorem 1.1.7 of [Hö].

References

[Hö] Lars Hörmander, *The Analysis of Linear Partial Differential Operators I*, Second Edition, 1990.

[HuHu] John H. Hubbard and Barbara Burke Hubbard, *Vector Calculus, Linear Algebra, and Differential Forms*, Third Edition, 2007.