Multivariable integration

These notes cover integrals of continuous functions of several real variables. They use iterated integration and differentiation to reduce to single variable results. Results are extended to general open sets using partitions of unity. Some similar results can be found in Spivak's *Calculus on Manifolds* and in other books on integration. The treatment of monotone convergence follows Riesz and Sz.-Nagy's *Functional Analysis* and even moreso Sz.-Nagy's *Introduction to Real Functions and Orthogonal Expansions*. Some good further references for differential forms include Shifrin's *Multivariable Mathematics*, Taylor's *Introduction to Analysis in Several Variables*, and do Carmo's *Differential Forms and Applications*.

1. Integration on products of intervals

Products of intervals are the easiest sets to integrate over.

1.1. Background on partial derivatives.

We begin with Leibniz' theorem on exchanging a derivative and an integral.

THEOREM 1.1.1. *If $f\colon [a,b]\times[c,d]\to\mathbb{R}$ is continuous, then so is $g\colon[a,b]\to\mathbb{R}$ given by $g(x)=\int_c^d f(x,y)\,dy$. If further $\partial_x f$ exists and is continuous on $[a,b]\times[c,d]$, then g is differentiable and*

$$g'(x)=\int_c^d \partial_x f(x,y)\,dy.$$

Proof. Since f is continuous on a compact set, it is uniformly continuous: given $\varepsilon>0$ there is $\delta>0$ such that $|h|<\delta$ implies $|f(x+h,y)-f(x,y)|<\varepsilon$ for all x and y. Hence if $|h|<\delta$ then

$$|g(x+h)-g(x)|=\Big|\int_c^d f(x+h,y)\,dy-\int_c^d f(x,y)\,dy\Big|\le\int_c^d \Big|f(x+h,y)-f(x,y)\Big|\,dy<\varepsilon(d-c).$$

Since $\varepsilon>0$ was arbitrary, this completes the proof of the first claim.

The proof of the second claim is similar. Given x, y, and h, by the mean value theorem there is x^* between x and $x+h$ such that $f(x+h,y)-f(x,y)=\partial_x f(x^*,y)h$. And given $\varepsilon>0$ there is $\delta>0$ such that $|h|<\delta$ implies $|\partial_x f(x^*,y)-\partial_x f(x,y)|<\varepsilon$. Hence if $|h|<\delta$ then

$$\Big|\int_c^d\Big(\frac{f(x+h,y)-f(x,y)}{h}-\partial_x f(x,y)\Big)\,dx\Big|<\varepsilon(d-c).$$

Since $\varepsilon>0$ was arbitrary, this completes the proof of the second claim. $\square$

We will also need Clairaut's theorem on exchanging partial derivatives.

THEOREM 1.1.2. *Let $F\colon\mathbb{R}^2\to\mathbb{R}$ be a function such that $\partial_x F$, $\partial_y F$, and $\partial_y\partial_x F$ all exist near a point (x,y), and such that $\partial_y\partial_x F$ is continuous at (x,y). Then $\partial_x\partial_y F$ exists and equals $\partial_y\partial_x F$.*

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Proof. Let $G(x,y,h,k) = F(x+h,y+k) - F(x,y+k) - F(x+h,y) + F(x,y)$. By definition,

$$\partial_x \partial_y F(x,y) = \lim_{h\to 0} \frac{\partial_y F(x+h,y) - \partial_y F(x,y)}{h} = \lim_{h\to 0}\lim_{k\to 0} \frac{G(x,y,h,k)}{hk}, \tag{1.1.3}$$

provided the limits exist. By the mean value theorem applied to the function $t \mapsto F(x+t,y+k) - F(x+t,y)$ on the interval between 0 and h, we see that there is $x^* = x^*(x,y,h,k)$ between x and $x+h$ such that

$$G(x,y,h,k) = [\partial_x F(x^*,y+k) - \partial_x F(x^*,y)]\, h.$$

Applying now the mean value theorem to the function $t \mapsto \partial_x F(x^*, y+t)$ on the interval between 0 and k gives $y^* = y^*(x,y,h,k)$ between y and $y+k$ such that

$$G(x,y,h,k) = \partial_y \partial_x F(x^*,y^*)hk.$$

Plugging that into (1.1.3) gives

$$\partial_x \partial_y F(x,y) = \lim_{h\to 0}\lim_{k\to 0} \partial_y \partial_x F(x^*,y^*) = \partial_y \partial_x F(x,y),$$

where for the second equals we used the continuity of $\partial_y \partial_x F(x^*,y)$. □

1.2. **Iterated integrals on rectangles.**

We define the integral of a continuous function over a rectangle in terms of its iterated integrals. The fact that these are equal is called *Fubini's theorem.*

THEOREM 1.2.1. *If* $f\colon [a,b]\times[c,d] \to \mathbb{R}$ *is continuous then*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{1.2.2}$$

We call the value in (1.2.2) the *integral of* f *over* $[a,b]\times[c,d]$ and write it as $\iint_{[a,b]\times[c,d]} f$.

Proof. Let

$$F(x,y) = \int_a^x \int_c^y f(s,t)\,dt\,ds.$$

By the fundamental theorem of calculus and Clairaut's theorem (the hypotheses of Theorem 1.1.2 follow from Theorem 1.1.1), we have

$$f(x,y) = \partial_y \partial_x F(x,y) = \partial_x \partial_y F(x,y). \tag{1.2.3}$$

Take $\int_a^b dx$ of $f(x,y) = \partial_x \partial_y F(x,y)$, and then take $\int_c^d dy$, to get

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = F(b,d).$$

Finally, plugging in the definition of F gives (1.2.2). □

1.3. **Generalizations.**

We are now ready to define the integral of a continuous function over any product of rectangles.

DEFINITION 1.3.1. Let $R = [a_1,b_1]\times\cdots\times[a_n,b_n] \subset \mathbb{R}^n$, and let $f\colon R \to \mathbb{R}$ be continuous. Put

$$\int_R f = \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} f(x)\,dx_1 \cdots dx_n = \int_{a_{\sigma(n)}}^{b_{\sigma(n)}} \cdots \int_{a_{\sigma(1)}}^{b_{\sigma(1)}} f(x)\,dx_{\sigma(1)} \cdots dx_{\sigma(n)},$$

where σ is any permutation (i.e. any bijection on the set $\{1,\ldots,n\}$). The value is independent of σ because any permutation is a finite number of transpositions (a transposition is a permutation given by swapping two elements and leaving the rest alone) away from the identity.

Our results on switching order of derivatives and on switching a derivative and an integral carry over directly to this more general setting; we only ever need to switch one pair at a time.

The corresponding results for functions with values in $\mathbb{R}^m$ follow because it is enough to check them component by component. In particular, this covers the case of functions with values in $\mathbb{C}$.

2. Integration on open sets.

2.1. **Definition of the integral in terms of a partition of unity.** We next use the above to define $\int_U f$ where $U \subset \mathbb{R}^n$ is open. This requires some more setup.

Definition 2.1.1. The *support* of f, denoted $\operatorname{supp} f$, is the closure of the set where f is not zero. This set 'supports' the function in the sense that we think of the function as resting on it.

If f is continuous and has compact support in U, then we define

$$\int_U f = \int_{\mathbb{R}^n} f. \tag{2.1.2}$$

The latter integral is defined by iterated integration, as in Definition 1.3.1; note that all the infinite integrals reduce to finite ones because f has bounded support.

But the above definition cannot be applied directly to interesting examples like the one where f is constant, because extending f to be zero outside of U leads to a discontinuous integrand. The following device will allow us to write such an f as a sum of suitable functions.

Definition 2.1.3. Let $U \subset \mathbb{R}^n$ be open. A continuous *partition of unity* of U is a family of continuous functions $\{\rho_j\}_{j\in\mathcal{J}}$ on U, where $\mathcal{J}$ is a countable indexing set, such that

(1) $0 \le \rho_j(x) \le 1$ for all $j \in \mathcal{J}$ and $x \in U$.
(2) Each ρ_j has compact support in U.
(3) For any compact $K \subset U$, there is a finite $\mathcal{J}' \subset \mathcal{J}$, such that, for all $x \in K$, we have

$$\sum_{j\in\mathcal{J}} \rho_j(x) = \sum_{j\in\mathcal{J}'} \rho_j(x) = 1. \tag{2.1.4}$$

The name 'partition of unity' comes from $\sum \rho_j = 1$.

Example 2.1.5. The simplest continuous partition of unity of $\mathbb{R}$ is obtained by taking $\mathcal{J} = \mathbb{Z}$ and $\rho_j(x) = \rho(x-j)$, where $\rho(x) = \max(1-|x|, 0)$. See Figure 1.

Exercise 2.1.6.

(1) Let the ρ_j be as in Example 2.1.5, and let $a < b$ be given. Give an expression in terms of floor and ceiling functions (recall that $\lfloor x \rfloor$ is the greatest integer $\le x$ and $\lceil x \rceil$ is the least integer $\ge x$) for the number of nonzero terms in the sum $\sum_{j=-\infty}^{\infty} \int_a^b \rho_j$.
(2) Evaluate the sum in part (1) by switching the sum and integral, justifying this using the fact that the sum terminates.
(3) Construct a continuous partition of unity of $(0,\infty)$ having the form $\rho_j(x) = \rho(2^j x)$, where ρ is a piecewise linear function supported in $[1/2, 2]$. Draw a sketch.
(4) With ρ as in part (3), construct a continuous partition of unity of $(0, 2)$, using the functions $\rho_j(x) = \rho(2^j x)$ for $j \ge 1$, the functions $\rho_j(x) = \rho_{|j|}(2-x)$ for $j \le -1$, and a suitable function ρ_0. Draw a sketch.

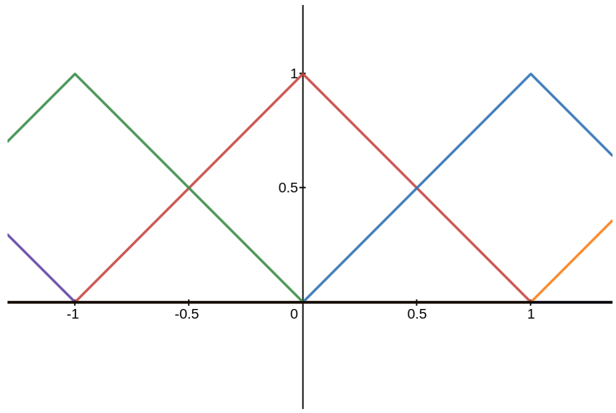

FIGURE 1. The partition of unity of Example 2.1.5. Each ρ_j is supported in $[j-1, j+1]$.

(5) Show that the following steps construct a C^∞ partition of unity (i.e. one in which the functions are C^∞ rather than just continuous) of $\mathbb{R}$: i) let[1] $f(x) = e^{-1/x}$ when $x > 0$ and $f(x) = 0$ when $x \le 0$, ii) let $\psi(x) = f(1-x)f(x+1)$, iii) let $\psi_j(x) = \psi(x-j)$, iv) let $\rho_j(x) = \psi_j(x)/\sum_{n\in\mathbb{Z}} \psi_n(x)$. Draw a sketch.

(6) Extend the above examples to higher dimensions, by letting $\rho_{j,k}(x,y) = \rho_j(x)\rho_k(y)$ for $\mathbb{R}^2$, $\rho_{j,k,\ell}(x,y) = \rho_j(x)\rho_k(y)\rho_\ell(z)$ for $\mathbb{R}^3$, etc.

Once we have constructed a partition of unity on an open set $U \subset \mathbb{R}^n$, we define, for continuous nonnegative f,

$$\int_U f = \sum_{j\in\mathcal{J}} \int_{\mathbb{R}^n} \rho_j f = \sup\Big\{ \sum_{j\in\mathcal{J}'} \int_{\mathbb{R}^n} \rho_j f \mid \mathcal{J}' \text{ a finite subset of } \mathcal{J} \Big\}, \tag{2.1.7}$$

where we allow the value $+\infty$. The point is that each $\int_{\mathbb{R}^n} \rho_j f$ can be defined and evaluated by (2.1.2). Moreover, any conceivable mode of summing over $\mathcal{J}$ can be used. The key to proving such a statement, and working with (2.1.7) more generally, is the following lemma.

LEMMA 2.1.8. *Let X and Y be subsets of $\mathbb{R}$. If for any x in X there exists y in Y such that $x \le y$, then* $\sup X \le \sup Y$.

Proof. The hypothesis implies that any upper bound of Y is an upper bound of X, which in turn implies $\sup X \le \sup Y$. □

EXAMPLE 2.1.9. If each $a_j \ge 0$, then

$$\lim_{J\to\infty} \sum_{j=-J}^{J} a_j = \sup_{J\in\mathbb{N}} \sum_{j=-J}^{J} a_j = \sup_{K\in\mathbb{N}} \sup_{L\in\mathbb{N}} \sum_{j=-K}^{L} a_j = \lim_{K\to\infty} \lim_{L\to\infty} \sum_{j=-K}^{L} a_j.$$

Proof. The first and last equalities follow from the fact that the sequences whose limits are being taken are nondecreasing. The middle equality follows in two steps. First, we have '$\le$' because $\{\sum_{j=-J}^{J} a_j \mid J \in \mathbb{N}\} \subset \{\sum_{j=-K}^{L} a_j \mid K,\ L \in \mathbb{N}\}$. Second, we have '$\ge$' by Lemma 2.1.8, because given L and K, we can get $\sum_{j=-K}^{L} a_j \le \sum_{j=-J}^{J} a_j$ by taking $J = \max(L, K)$. □

[1] *Hint:* Prove that f is C^∞ by proving that the set {functions g such that $g(t) = P(1/t)f(t)$ for some polynomial P} consists only of continuous functions and is closed under differentiation.

EXERCISE 2.1.10. Prove that if $a_{j,k} \geq 0$ for all $j \in \mathbb{N}$ and $k \in \mathbb{N}$, then

$$A = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} a_{j,k}, \qquad B = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} a_{j,k}, \qquad C = \sup \Big\{ \sum_{(j,k) \in \mathcal{F}} a_{j,k} \mid \mathcal{F} \text{ a finite subset of } \mathbb{N} \times \mathbb{N} \Big\},$$

are all equal.[2]

When U is an interval in $\mathbb{R}$, the definition in (2.1.7) reduces to a usual notion of a (proper or improper) integral over an interval. This is explained in the following lemma and exercise.

LEMMA 2.1.11. *Let (a, b) be an open interval in $\mathbb{R}$, bounded or unbounded, and let $f : (a, b) \to [0, \infty)$ be continuous. Then*

$$\int_{(a,b)} f = \sup \Big\{ \int_A^B f \mid [A, B] \text{ a compact subinterval of } (a, b) \Big\},$$

where we allow the value $+\infty$. More precisely, $\int_{(a,b)} f$ is defined by (2.1.7), and $\int_A^B f$ is the familiar single variable Riemann integral.

Proof. We first prove '$\geq$', and then prove '$\leq$', both by checking the hypothesis of Lemma 2.1.8.

1) Given $[A, B] \subset (a, b)$, by property (3) of Definition 2.1.3 there is a finite $\mathcal{J}' \subset \mathcal{J}$ such that $\sum_{j \in \mathcal{J}'} \rho_j = 1$ on $[A, B]$. That gives

$$\int_A^B f = \int_A^B \sum_{j \in \mathcal{J}'} \rho_j f = \sum_{j \in \mathcal{J}'} \int_A^B \rho_j f \leq \sum_{j \in \mathcal{J}'} \int_{\mathbb{R}} \rho_j f.$$

2) Given $\mathcal{J}' \subset \mathcal{J}$, by property (2) of Definition 2.1.3 there is $[A, B]$ containing the support of ρ_j for all $j \in \mathcal{J}'$. That gives

$$\sum_{j \in \mathcal{J}'} \int_{\mathbb{R}} \rho_j f = \int_{\mathbb{R}} \sum_{j \in \mathcal{J}'} \rho_j = \int_A^B \sum_{j \in \mathcal{J}'} \rho_j f \leq \int_A^B f,$$

where for the inequality we used properties (1) and (3) of Definition 2.1.3 . $\square$

EXERCISE 2.1.12. Let $F : (a, b) \times (a, b) \to \mathbb{R}$. Prove that if $F(x, y)$ is nonincreasing with respect to x and nondecreasing with respect to y, then

$$\sup \big\{ F(A, B) \mid [A, B] \text{ a compact subinterval of } (a, b) \big\} = \lim_{A \to a^+} \lim_{B \to b^-} F(A, B).$$

Now, let us check that the definition in (2.1.7) is independent of the partition of unity.

LEMMA 2.1.13. *Let $U \subset \mathbb{R}^n$ be open, let $f : U \to [0, \infty)$ be continuous, and let $\{\rho_j\}_{j \in \mathcal{J}}$ and $\{\tau_\ell\}_{\ell \in \mathcal{L}}$ be continuous partitions of unity on U. Then*

$$\sum_{j \in \mathcal{J}} \int_{\mathbb{R}^n} \rho_j f = \sum_{\ell \in \mathcal{L}} \int_{\mathbb{R}^n} \tau_\ell f,$$

where we allow the value $+\infty$.

[2]*Hint:* Prove $A = C$ as in Example 2.1.9. Then deduce $B = C$ without any extra work.

Proof. By symmetry, it is enough to prove '$\leq$'. Given $\mathcal{J}' \subset \mathcal{J}$ finite, take $\mathcal{L}' \subset \mathcal{L}$ finite such that $\sum_{\ell \in \mathcal{L}'} \tau_\ell = 1$ on $\operatorname{supp} \rho_j$ for each $j \in \mathcal{J}'$. Then

$$\sum_{j \in \mathcal{J}'} \int_{\mathbb{R}^n} \rho_j f = \sum_{j \in \mathcal{J}'} \int_{\mathbb{R}^n} \sum_{\ell \in \mathcal{L}'} \tau_\ell \rho_j f = \sum_{\ell \in \mathcal{L}'} \int_{\mathbb{R}^n} \sum_{j \in \mathcal{J}'} \tau_\ell \rho_j f \leq \sum_{\ell \in \mathcal{L}'} \int_{\mathbb{R}^n} \tau_\ell f.$$

The conclusion now follows from Lemma 2.1.8. □

2.2. **Constructing a partition of unity.**

THEOREM 2.2.1. *Let $U \subset \mathbb{R}^n$ be open. Then there exists a partition of unity on U.*

Proof. Let $B_r(x) = \{y \in \mathbb{R}^n \colon |y - x| < r\}$. We define the partition of unity $\rho_1,\ \rho_2, \dots$ by

$$\psi(x) = \max(0, 1 - |x|), \qquad \psi_j(x) = \psi\Big(\frac{x - x_j}{r(x_j)}\Big), \qquad \rho_j(x) = \frac{\psi_j(x)}{\sum_{m=1}^\infty \psi_m(x)}, \tag{2.2.2}$$

where $x_1,\ x_2, \dots \in U$ and $r(x_1),\ r(x_2), \dots > 0$ will be defined below in such a way that

$$U = \bigcup_{j=1}^\infty B_{r(x_j)}(x_j), \tag{2.2.3}$$

and for every compact $K \subset U$ there is $J(K) \in \mathbb{N}$ such that

$$j \geq J(K) \quad \Longrightarrow \quad \overline{B_{r(x_j)}(x_j)} \cap K = \varnothing. \tag{2.2.4}$$

We define the x_j and $r(x_j)$in two steps.

1. Construct a family of compact sets A_ℓ which cover U with small overlap. To define them, let $U_0 = \varnothing$, and for $\ell \in \mathbb{N}$ let

$$U_\ell = \{x \in U \mid |x| < \ell \text{ and } \overline{B_{1/\ell}(x)} \subset U\}, \qquad A_\ell = \overline{U_\ell} \setminus U_{\ell - 1}.$$

Thus

$$U = \bigcup_{\ell=1}^\infty U_\ell = \bigcup_{\ell=1}^\infty A_\ell. \tag{2.2.5}$$

2. Cover each A_ℓ by a finite family of open balls small enough that their closures are disjoint from $A_{\ell'}$ for $\ell' \leq \ell - 2$. To do this, for each $x \in U$, let $\ell(x) = \min\{\ell \in \mathbb{N} \mid x \in \overline{U_\ell}\}$, and take $r(x) > 0$ such that $\overline{B_{r(x)}(x)} \subset U \setminus \overline{U_{\ell(x)-1}}$. For each ℓ, by compactness of A_ℓ, there exists a finite set of points $x_{\ell,1}, \dots, x_{\ell, M_\ell} \in A_\ell$ such that

$$A_\ell \subset \bigcup_{m=1}^{M_\ell} B_{r(x_{\ell,m})}(x_{\ell,m}). \tag{2.2.6}$$

Put together those finite sets of points into a sequence $x_1,\, x_2\, \dots$.

Then (2.2.3) follows from (2.2.5) and (2.2.6). Given a compact $K \subset U$, fix $L \in \mathbb{N}$ such that $K \subset U_L$. Then (2.2.4) follows from the fact that $\overline{B_{r(x_{\ell,m})}(x_{\ell,m})} \cap K = \varnothing$ for $\ell \geq L + 2$. □

EXERCISE 2.2.7.

(1) Prove that (2.2.2) defines a partition of unity on U. Where are (2.2.3) and (2.2.4) used?
(2) Expand on the last two sentences of the proof of Theorem 2.2.1 by explaining why there exists $L \in \mathbb{N}$ such that $K \subset U_L$ and why $\overline{B_{r(x_{\ell,m})}(x_{\ell,m})} \cap K = \varnothing$ for $\ell \geq L + 2$.

(3) Show that the proof of Theorem 2.2.1 can be modified to yield a C^∞ partition of unity (i.e. one in which the functions are C^∞ rather than just continuous) of U by replacing the definition of ψ with $\psi(x) = \exp(\frac{1}{|x|^2-1})$ when $|x| < 1$ and $\psi(x) = 0$ when $|x| \geq 1$.

(4) Let $\mathcal{O}$ be an open cover of U. A partition of unity is said to be *subordinate to* $\mathcal{O}$ if, for every j, there is $O \in \mathcal{O}$ such that ρ_j is supported in O. Show that the proof of Theorem 2.2.1 can be modified to yield a partition of unity subordinate to a given open cover $\mathcal{O}$ by replacing the condition $\overline{B(x, r(x))} \subset U \setminus \overline{U_{\ell(x)-1}}$ with $\overline{B(x, r(x))} \subset O(x) \cap U \setminus \overline{U_{\ell(x)-1}}$, where $O(x)$ is any element of $\mathcal{O}$ such that $x \in O(x)$.

2.3. **Monotone convergence theorems.** Given $U \subset \mathbb{R}^n$ open, $f\colon U \to [0,\infty)$ continuous, and $\{\rho_j\}_{j=1}^\infty$ a partition of unity on U, we have defined

$$\int_U f = \sum_{j=1}^\infty \int_{\mathbb{R}^n} \rho_j f = \sum_{j=1}^\infty \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \rho_j f \, dx_n \cdots dx_1.$$

Next we wish to simplify this by writing it without the partition of unity, as

$$\int_U f = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f \mathbf{1}_U \, dx_n \cdots dx_1, \tag{2.3.1}$$

where $\mathbf{1}_U$ is the *characteristic function* or *indicator function* of U, given by

$$\mathbf{1}_U(x) = \begin{cases} 1, & x \in U, \\ 0, & x \notin U. \end{cases} \tag{2.3.2}$$

This is the same as showing that

$$\lim_{N\to\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_N \, dx_n \cdots dx_1 = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \lim_{N\to\infty} f_N \, dx_n \cdots dx_1,$$

where $f_N = \sum_{j=1}^N \rho_j f$. A statement of this kind is called a 'monotone convergence theorem', because the f_N tend monotonically to the limit $f\mathbf{1}_U$.

To prove a suitable monotone convergence theorem, we extend our notion of integral using the following two definitions.

Definition 2.3.3. A *step function* is a function $\varphi\colon \mathbb{R} \to \mathbb{R}$ which has a constant value c_k on each of a finite set of m intervals (x_{k-1}, x_k), with $x_0 < x_1 < \cdots < x_m$, and which is zero outside of $[x_0, x_m]$. At the endpoints of the intervals the function may be defined arbitrarily. In other words, a step function is a linear combination of finitely many indicator functions of bounded intervals and points. See Figure 2 for an example.

The integral of φ is given by

$$\int_{\mathbb{R}} \varphi = \sum_{k=1}^m c_k (x_k - x_{k-1}). \tag{2.3.4}$$

Exercise 2.3.5. Let φ and ψ be step functions. Use (2.3.4) to prove the following basic properties of their integrals.[3]

[3] *Hint:* Let $\{x_0, \ldots, x_m\}$ be the endpoints of the intervals for φ, and let $\{y_0, \ldots, y_\ell\}$ be the endpoints of the intervals for ψ. The difficulty is that φ might not have a constant value on each interval (y_{k-1}, y_k) and ψ might not have a constant value on each interval (x_{k-1}, x_k). To deal with this, put $\{z_0, \ldots, z_M\} = \{x_0, \ldots, x_m\} \cup \{y_0, \ldots, y_\ell\}$, and observe that each of φ and ψ has a constant value on each interval (z_{k-1}, z_k).

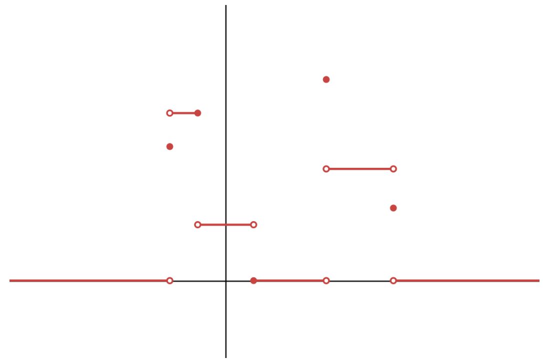

FIGURE 2. An example of a nonnegative step function with $m = 4$.

(1) If α and β are real numbers, then
$$\int_{\mathbb{R}} (\alpha\varphi + \beta\psi) = \alpha \int_{\mathbb{R}} \varphi + \beta \int_{\mathbb{R}} \psi.$$
(2) If $\varphi(x) \leq \psi(x)$ for all x, then $\int_{\mathbb{R}} \varphi \leq \int_{\mathbb{R}} \psi$.

DEFINITION 2.3.6. A function $f\colon \mathbb{R} \to \mathbb{R}$ is a *limit from below of step functions* if there exists a sequence of step functions $\varphi_1 \leq \varphi_2 \leq \cdots$ such that $\varphi_n(x) \to f(x)$ for all $x \in \mathbb{R}$. For such f we define
$$\int_{\mathbb{R}} f = \lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n. \tag{2.3.7}$$

EXAMPLE 2.3.8. Let $E \subset \mathbb{R}$ be countable. The the indicator function $\mathbf{1}_E$ (defined as in (2.3.2)) is a limit from below of step functions and $\int_{\mathbb{R}} \mathbf{1}_E = 0$. To show this, let $p_1,\ p_2, \ldots$ be an enumeration of E, and let $\varphi_n(x) = 1$ if $x = p_j$ for some $j \leq n$ and $\varphi_n(x) = 0$ otherwise. The most famous case is *Dirichlet's function* $\mathbf{1}_{\mathbb{Q}}$.

We must check that the limit in (2.3.7) is independent of the choice of the φ_n. The main step is the following surprisingly tricky lemma.

LEMMA 2.3.9. *Let $\varphi_1 \geq \varphi_2 \geq \cdots$ be a sequence of step functions such that $\lim_{n\to\infty} \varphi(x) = 0$ for all x. Then*
$$\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n = 0.$$

In a way the lemma is simple, because there are only two basic examples to keep in mind of monotonic convergence to zero for a sequence of step functions. The first is support not going to zero but maxima going to zero, as in $\varphi_n = \mathbf{1}_{[0,1]}/n$. The second is maxima not going to zero but support going to zero, as in $\varphi_n = \mathbf{1}_{[0,1/n]}$. The trickiness comes in from the fact that a general sequence may combine elements of these two basic examples in complicated ways.

Proof. Let $[a,b]$ be a compact interval containing the support of φ_1. Let $E = \{a_k\}_{k=1}^{\infty}$ be the set of all points where at least one of the φ_n is not continuous. Thus $E \subset [a,b]$ and if $x \notin [a,b]$ then $\varphi_n(x) = 0$ for all n.

Let $\varepsilon > 0$ be given. For every $p \notin E$ we have $\lim_{n\to\infty} \varphi_n(p) = 0$, so there is $N_p \in \mathbb{N}$ such that
$$\varphi_{N_p}(p) < \varepsilon.$$

Since φ_{N_p} is constant near p, there is an open interval I_p containing p such that $\varphi_{N_p}(x) < \varepsilon$ for x in I_p. Since $\varphi_1 \geq \varphi_2 \geq \cdots$, we obtain

$$\varphi_n(x) < \varepsilon, \qquad \text{when } n \geq N_p \text{ and } x \in I_p.$$

Let $\mathcal{I} = \{I_p\}_{p \in [a,b] \setminus E}$. Then $\mathcal{I}$ is an open cover of $[a, b] \setminus E$, and we supplement it with a cover of E by putting $\mathcal{J} = \{J_k\}_{k=1}^{\infty}$, where $J_k = (a_k - \varepsilon 2^{-k}, a_k + \varepsilon 2^{-k})$. Thus $\mathcal{J}$ is a collection of open intervals covering E and having total length $< 2\varepsilon$, and $\mathcal{I} \cup \mathcal{J}$ is an open cover of $[a, b]$, so by compactness it has a finite subcover $\{I_{p_1}, \ldots, I_{p_\ell}, J_{k_1}, \ldots, J_{k_m}\}$.

Let $N = \max\{N_{p_1}, \ldots, N_{p_\ell}\}$, so that

$$\varphi_n(x) < \varepsilon, \qquad \text{when } n \geq N \text{ and } x \in I_{p_1} \cup \cdots \cup I_{p_\ell}.$$

Hence

$$\int_{I_{p_1} \cup \cdots \cup I_{p_\ell}} \varphi_n < \varepsilon(b - a), \qquad \text{when } n \geq N.$$

The rest of $[a, b]$ is covered by $\{J_{k_1}, \ldots, J_{k_m}\}$, and these intervals have total length $< 2\varepsilon$. Since $\varphi_n(x) \leq \varphi_1(x) \leq \max \varphi_1$, we get

$$\int_{J_{k_1} \cup \cdots \cup J_{k_m}} \varphi_n < 2\varepsilon \max \varphi_1.$$

Thus, if $n \geq N$, then

$$0 \leq \int_{\mathbb{R}} \varphi_n \leq \int_{I_{p_1} \cup \cdots \cup I_{p_\ell}} \varphi_n + \int_{J_{k_1} \cup \cdots \cup J_{k_m}} \varphi_n < \varepsilon(b - a + 2 \max \varphi_1).$$

Since $\varepsilon > 0$ was arbitary, this completes the proof. □

Corollary 2.3.10. *Let $\varphi_1 \geq \varphi_2 \geq \cdots$ be a sequence of step functions such that* $\lim_{n \to \infty} \varphi(x) \leq 0$ *for all x. Then*

$$\lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n \leq 0.$$

Proof. Apply Lemma 2.3.9 to the sequence of step functions $\max(0, \varphi_1), \max(0, \varphi_2), \ldots$. □

Now we are ready to prove the following result, which implies that the limit in (2.3.7) is independent of the choice of φ_n, and also has other important consequences

Theorem 2.3.11. *Let $\varphi_1 \leq \varphi_2 \leq \cdots$ and $\psi_1 \leq \psi_2 \leq \cdots$ be sequences of step functions, such that $\varphi_n(x) \to f(x)$ and $\psi_n(x) \to g(x)$ for all x. If $f(x) \leq g(x)$ for all x, then*

$$\lim_{n \to \infty} \int_{\mathbb{R}} \varphi_n \leq \lim_{n \to \infty} \int_{\mathbb{R}} \psi_n.$$

Exercise 2.3.12. Complete the proof of Theorem 2.3.11 by filling in the following steps:

(1) Show that, for any m, the sequence $\varphi_m - \psi_1$, $\varphi_m - \psi_2$, $\ldots$ obeys the hypotheses of Corollary 2.3.10.
(2) Use Corollary 2.3.10 to show that $\int_{\mathbb{R}} \varphi_m \leq \lim_{n \to \infty} \int_{\mathbb{R}} \psi_n$.
(3) Take $m \to \infty$ of both sides of the result of part (2).

Exercise 2.3.13. Show that Theorem 2.3.11 implies the following:

(1) If $\varphi_1 \le \varphi_2 \le \cdots$ and $\psi_1 \le \psi_2 \le \cdots$ are sequences of step functions obeying $\lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \psi_n(x)$ for all x, then $\lim_{n\to\infty} \int_{\mathbb{R}} \varphi_n = \lim_{n\to\infty} \int_{\mathbb{R}} \psi_n$. Hence the limit in (2.3.7) is independent of the choice of sequence.
(2) If f and g are limits from below of step functions, then so is $f+g$, and $\int_{\mathbb{R}}(f+g) = \int_{\mathbb{R}} f + \int_{\mathbb{R}} g$.
(3) If $f\colon \mathbb{R} \to [0,\infty)$ is a limit from below of step functions and $\int_{\mathbb{R}} f = 0$, then $f(x) = 0$ for all but countably many x.[4]
(4) If $f\colon \mathbb{R} \to [0,\infty)$ is a limit from below of step functions, then

$$\int_{\mathbb{R}} f = \sup\Big\{ \int_{\mathbb{R}} \psi \colon \psi \text{ is a step function with } \psi \le f \Big\}.^{5}$$

We next verify that (2.3.7) recovers the familiar Riemann integral when f is continuous.

Proposition 2.3.14. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous on $[a,b]$ and zero outside of $[a,b]$. Then $\int_{\mathbb{R}} f$ as defined by (2.3.7) equals the Riemann integral of f on $[a,b]$.*

Proof. For each n, let $a = x_0 < x_1 < \cdots < x_{2^n} = b$ be equally spaced points in $[a,b]$, and define the step function φ_n to have the constant the constant value $c_k = \min_{[x_{k-1},x_k]} f$ on each (x_{k-1}, x_k), and $\varphi_n(x_k) = f(x_k)$ for all k, and $\varphi_n(x) = 0$ for x not in $[a,b]$. The integral $\int_{\mathbb{R}} \varphi_n$ is the lower Riemann sum for the partition of the interval $[a.b]$ into 2^n equal subintervals: see Figure 3.

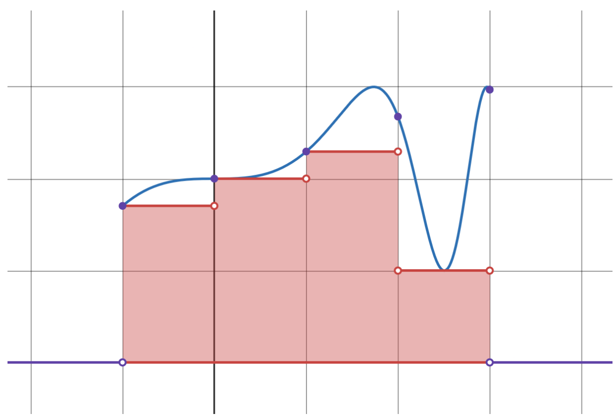

Figure 3. The blue graph is f, the red is φ_2, and the purple is the values they share. The shaded red region, $\int_{\mathbb{R}} \varphi_2$, has area equal to the lower Riemann sum corresponding to $n = 2$.

Since $\varphi_n(x) \to f(x)$ for every x, we have $\int_{\mathbb{R}} f = \int_{\mathbb{R}} \varphi_n$. On the other hand $\int_{\mathbb{R}} \varphi_n$ converges to the Riemann integral of f on $[a,b]$ because the mesh of the partition is going to zero. ☐

Exercise 2.3.15. Prove that, in the setting of the proof of Proposition 2.3.14, $\varphi_n \to f$ uniformly on $[a,b]$ by showing that for every ε there exists N such that $0 \le f(x) - \varphi_n(x) < \varepsilon$ when $n \ge N$.

We are now ready to prove our needed monotone convergence theorem:

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[4] *Hint:* First prove that if a step function is nonnegative and integrates to zero, then it vanishes off of a finite set. Then let $\varphi_1 \le \varphi_2 \le \cdots$ be a sequence of step functions converging to f pointwise, and let $\psi_n = \max(0, \varphi_n)$.

[5] *Hint:* To show it is an upper bound, given ψ let $\psi_n = \max(\varphi_n, \psi)$

Theorem 2.3.16 (Monotone Convergence Theorem). *Let $h_1 \le h_2 \le \cdots$ be functions which are each limits from below of step functions, and suppose these functions converge pointwise to h. Then h is a limit from below of step functions, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} h_n = \int_{\mathbb{R}} h.$$

Proof of Theorem 2.3.16. By definition there are sequences of step functions

$$\begin{aligned}&\varphi_{1,1} \le \varphi_{1,2} \le \cdots \to h_1,\\ &\varphi_{2,1} \le \varphi_{2,2} \le \cdots \to h_2,\\ &\text{etc.,}\end{aligned}$$

Define a new sequence of step functions $\varphi_n = \max\{\varphi_{j,k} \mid 1 \le j,k \le n\}$. Then $\varphi_1 \le \varphi_2 \le \cdots$ and

$$\varphi_{m,n} \le \varphi_n \le h, \text{ when } m \le n.$$

Letting $n \to \infty$ shows $h_m \le \lim \varphi_n \le h$, and then letting $m \to \infty$ shows $\lim \varphi_n = h$.

On the other hand, $\varphi_n \le h_n \le h$ for each n, so

$$\int_{\mathbb{R}} \varphi_n \le \int_{\mathbb{R}} h_n \le \int_{\mathbb{R}} h.$$

Since $\int h = \lim \int \varphi_n$ by definition, we obtain $\int h = \lim \int h_n$. □

We now conclude (2.3.1) by putting $f_N = \sum_{j=1}^N \rho_j f$ and writing

$$\begin{aligned}\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f \mathbf{1}_U \, dx_n \cdots dx_1 &= \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \lim_{N\to\infty} f_N \, dx_n \cdots dx_1 = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \lim_{N\to\infty} \Big(\int_{\mathbb{R}} f_N \, dx_n \Big) x_{n-1} \cdots dx_1 \\ &= \cdots = \lim_{N\to\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_N \, dx_n \cdots dx_1 = \sum_{j=1}^{\infty} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \rho_j f \, dx_n \cdots dx_1.\end{aligned}$$

Exercise 2.3.17. Monotone convergence theorems hold in more general settings than the one discussed here. An interesting limitation on them is provided by the following example due to Vitali. To set it up, define an equivalence relation on $\mathbb{R}$ by $x \sim y$ when $x - y \in \mathbb{Q}$. Let A be a subset of $[0,1]$ containing exactly one representative from each equivalence class.[6] Let $p_1, p_2, \ldots$ be an enumeration of the rational numbers in $[-1,1]$, Let

$$f_n(x) = \mathbf{1}_A(x - p_n), \qquad g_N = \sum_{n=1}^N f_n.$$

Suppose we have a definition of $\int_{\mathbb{R}}$ such that $\mathbf{1}_A$ is integrable, and such that for all integrable functions f and g and for all real numbers a and b we have

$$f \le g \Rightarrow \int_{\mathbb{R}} f \le \int_{\mathbb{R}} g, \quad \int_{\mathbb{R}} (af + bg) = a \int_{\mathbb{R}} f + b \int_{\mathbb{R}} g, \quad \int_{\mathbb{R}} f(x) dx = \int_{\mathbb{R}} f(x-a) dx. \tag{2.3.18}$$

Suppose moreover that if $[a,b]$ is a compact interval then $\int_{\mathbb{R}} \mathbf{1}_{[a,b]} = b - a$.

(1) Prove that the relation $\sim$ defined above is an equivalence relation (i.e. that it is reflexive, symmetric, and transitive).

[6] Such a subset exists by the Axiom of Choice.

(2) Prove that if $\mathbf{1}_A$ is integrable, then $\int_{\mathbb{R}} g_N = \int_{\mathbb{R}} \mathbf{1}_A = 0$ for all N, by showing that $g_N \le \mathbf{1}_I$ for a suitable interval I.
(3) Prove that there exists a function g such that, for all x, $g_N(x) \to g(x)$ as $N \to \infty$.
(4) By showing that $g \ge \mathbf{1}_J$ for a suitable interval J, show that if g is integrable then $\int_{\mathbb{R}} g > 0$, and hence that $\lim \int_{\mathbb{R}} g_N < \int_{\mathbb{R}} \lim g_N$.[7]

Exercise 2.3.19. The list of properties (2.3.18) brings to light a deficiency in our concept of 'limit from below of step functions', which can be observed and remedied as follows.

(1) Prove that $\mathbf{1}_{(\mathbb{R}\setminus\mathbb{Q})\cap[0,1]}$ is not a limit from below of step functions, and hence that the set of limits from below of step functions is not a vector space.
(2) To remedy this, define h to be a 'limit from above of step functions' if $-h$ is a limit from below of step functions. If $f = g + h$, where g is a limit from below of step functions and h is a limit from above of step functions, define $\int_{\mathbb{R}} f = \int_{\mathbb{R}} g + \int_{\mathbb{R}} h$. Prove that this definition is independent of the choice of g and h.[8]
(3) Verify the properties (2.3.18) for this new expanded notion of integral.[9]

2.4. **Vector-valued integrands.** We have now defined $\int_U f$, whenever $U \subset \mathbb{R}^n$ is open and $f\colon U \to [0,\infty)$ is continuous. Note that we then have $0 \le \int_U f \le +\infty$. This includes the important fundamental case of $f = \mathbf{1}_U$, in which case we write

$$\int_U 1 = \operatorname{vol} U,$$

with $\operatorname{vol} U$ denoting the n-dimensional volume of U.

We extend this notion of integral in several steps.

If $f\colon U \to \mathbb{R}$ is continuous, we put $f_+ = \max(0,f)$ and $f_- = \max(0,-f)$. Then $f = f_+ - f_-$. We then define

$$\int_U f = \int_U f_+ - \int_U f_-,$$

where we now allow both the value $+\infty$ and the value $-\infty$ *unless* $\int_U f_+ = \int_U f_- = +\infty$; in this last case we leave $\int_U f$ undefined due to the ambiguity of $+\infty - \infty$. If $\int_U f_+ < +\infty$ and $\int_U f_- < +\infty$ then $\int_U f$ is a real number (and not $\pm\infty$) and we say $\int_U f$ *converges absolutely.*

If $f\colon U \to \mathbb{R}^m$ is continuous, we write $f = (f_1, \dots f_m)$ and define

$$\int_U f = \Big(\int_U f_1, \cdots \int_U f_m\Big),$$

provided each $\int_U f_j$ converges absolutely.

If $f\colon U \to \mathbb{C}$ is continuous, we define $\int_U f$ by identifying $\mathbb{C}$ with $\mathbb{R}^d$ and using the case above.

[7] One says that the area under the graph of $\mathbf{1}_A$ is finitely additive but not countably additive and that A is Banach measurable but not Lebesgue measurable. In dimension ≥ 3 even finite additivity can break down as shown by the Banach–Tarski paradox.

[8] *Hint:* Write $g_1 + h_1 = g_2 + h_2$ as $g_1 - h_2 = g_2 - h_1$ and using part (2) of Exercise 2.3.13.

[9] With this expanded notion we are very close to the definition of Lebesgue integration. Indeed the only thing missing is to weaken 'for all $x \in \mathbb{R}$' to 'for almost all $x \in \mathbb{R}$' in Definition 2.3.6. A beautiful presentation of the details of this can be found in Section 5.1 of Sz.-Nagy's *Introduction to Real Functions and Orthogonal Expansions.*

3. Change of variables

3.1. **Single variable version.** The change of variables theorem in one dimension says that if $f\colon [A,B] \to \mathbb{R}$ is continuous and $\varphi\colon [a,b] \to [A,B]$ is C^1, and if $\varphi(a) = A$ and $\varphi(b) = B$, then

$$\int_a^b f(\varphi(x))\varphi'(x)\,dx = \int_A^B f(y)\,dy.$$

We prove this using the fundamental theorem of calculus: let $h(y) = \int_A^y f(z)\,dz$. Then $h'(y) = f(y)$ and the right hand side becomes $h(B)$ while the left hand side becomes

$$\int_a^b h'(\varphi(x))\varphi'(x)\,dx = \int_a^b \frac{d}{dx}h(\varphi(x))\,dx = h(\varphi(b)) - h(\varphi(a)) = h(B) - h(A) = h(B).$$

We combine this with the case where $\varphi(a) = B$ and $\varphi(b) = A$ by writing this as

$$\int_I f(\varphi(x))|\varphi'(x)|\,dx = \int_{\varphi(I)} f(y)\,dy,$$

when I is an open interval and $\varphi' \neq 0$ on I, or as

$$\int_{\mathbb{R}} f(\varphi(x))|\varphi'(x)|\,dx = \int_{\mathbb{R}} f(y)\,dy, \tag{3.1.1}$$

provided $\varphi\colon I \to \mathbb{R}$ has nowhere vanishing derivative and the support of f is contained in $\varphi(I)$.

Below we gradually build up to a higher dimensional version.

3.2. **Linear change of variables.** We begin with the simplest changes of variables, and assume $f\colon \mathbb{R}^n \to \mathbb{R}$ continuous and compactly supported. The very simplest case is translation: if $\varphi(x) = x + b$, then

$$\int_{\mathbb{R}^n} f(\varphi(x))\,dx = \int_{\mathbb{R}^n} f, \tag{3.2.1}$$

because $\int_{\mathbb{R}} h(x_j + b_j)dx_j = \int_{\mathbb{R}} h(x_j)dx_j$ by (3.1.1).

Next, suppose $\varphi(x) = Ex$, where E is an elementary matrix. There are three kinds:

- If φ switches two entries of x, then we have (3.2.1) because
$$\int_{\mathbb{R}}\int_{\mathbb{R}} h(x_j, x_k)dx_j dx_k = \int_{\mathbb{R}}\int_{\mathbb{R}} h(x_k, x_j)dx_j dx_k$$
by Fubini's theorem (1.2.2).
- If φ multiplies one entry of x by a scalar c, then we have
$$|c|\int_{\mathbb{R}^n} f(\varphi(x))\,dx = \int_{\mathbb{R}^n} f, \tag{3.2.2}$$
because $\int_{\mathbb{R}} h(cx)|c|dx = \int_{\mathbb{R}} h(x)dx$ by (3.1.1).
- If φ adds a multiple of one entry to another entry, then we get (3.2.1) again. To see this, write the mapping as $\varphi(x_1, \dots, x_j, \dots, x_n) = (x_1, \dots, x_j + cx_k, \dots, x_n)$, and then write the iterated integral with x_j on the inside:
$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(\varphi(x))\,dx_j \cdots dx_n\,dx_1 \cdots dx_{j-1} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f(x)\,dx_j \cdots dx_n\,dx_1 \cdots dx_{j-1} = \int_U f,$$
again using $\int_{\mathbb{R}} h(x_j + b_j)dx_j = \int_{\mathbb{R}} h(x_j)dx_j$ by (3.1.1).

Now recall that any invertible matrix is a product of elementary matrices (the row operations that bring it to reduced row echelon form), that $\det(AB) = \det A \det B$, and that the determinants of the three kinds of elementary matrices are respectively -1, c, and 1. Those facts give

$$|\det A| \int_{\mathbb{R}^n} f(\varphi(x))\,dx = \int_{\mathbb{R}^n} f.$$

Thus we have proved that

$$\int_{\mathbb{R}^n} f(\varphi(x))|\det \varphi'(x)|\,dx = \int_{\mathbb{R}^n} f, \tag{3.2.3}$$

when $\varphi(x) = Ax + b$ with A invertible and $f \colon \mathbb{R}^n \to \mathbb{R}$ continuous and compactly supported, and we will generalize this formula below.

Definition 3.2.4. Let U and V be open subsets of $\mathbb{R}^n$. A function $\varphi \colon V \to U$ is a *diffeomorphism* if it is bijective and both φ and φ^{-1} are C^∞.

Exercise 3.2.5.

(1) Let U and V be open subsets of $\mathbb{R}^n$ and $\varphi \colon V \to U$ a diffeomorphism. Suppose that (3.2.3) holds for that φ and for all $f \colon U \to \mathbb{R}$ continuous and compactly supported. Prove that

$$\int_V f(\varphi(x))|\det \varphi'(x)|\,dx = \int_U f$$

for all continuous $f \colon U \to [0, \infty)$.[10]

(2) Use the result of part (1) to show that if $\varphi(x) = Ax + b$ with A invertible, then $\operatorname{vol} U = |\det A| \operatorname{vol} V$, where the value $+\infty$ is allowed.

(3) Use the result of part (2) to show that the volume of a parallelepiped in $\mathbb{R}^n$ spanned by a set of vectors $\{v_j\}_{j=1}^n$ equals $|\det A|$, where A is any matrix whose columns are the v_j.

3.3. **Differential forms.** To handle more general changes of variables it is helpful to use the algebra of differential forms, following Lax's 1998 paper *Change of Variables in Multiple Integrals* and Section 4.5 of Taylor's book *Introduction to Analysis in Several Variables*. The key properties are as follows.

(1) If $v_1, \ldots, v_k$ are vectors in $\mathbb{R}^n$, and the components of each v_j are $v_{j,1}, \ldots, v_{j,n}$, then

$$(dx_{j_1} \wedge \cdots \wedge dx_{j_k})(v_1, \ldots, v_k) = \det \begin{pmatrix} v_{1,j_1} & \cdots & v_{k,j_1} \\ \vdots & \ddots & \vdots \\ v_{1,j_k} & \cdots & v_{k,j_k} \end{pmatrix}.$$

(2) If $f \colon \mathbb{R}^n \to \mathbb{R}$ is smooth, then

$$df = \partial_{x_1} f dx_1 + \cdots + \partial_{x_n} f dx_n, \tag{3.3.1}$$

and

$$d(f dx_{j_1} \wedge \cdots \wedge dx_{j_k}) = df \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_k}.$$

(3) If $f \colon \mathbb{R}^n \to \mathbb{R}$ and $\varphi \colon \mathbb{R}^m \to \mathbb{R}^n$ are smooth, then

$$\varphi^*(f(y)dy_{j_1} \wedge \cdots \wedge dy_{j_k}) = f(\varphi(x))d(y_{j_1} \circ \varphi) \wedge \cdots \wedge d(y_{j_k} \circ \varphi) = f(\varphi(x))d\varphi_{j_1} \wedge \cdots \wedge d\varphi_{j_k}.$$

If $m = k = n$, then

$$\varphi^*(f(y)dy_1 \wedge \cdots \wedge dy_n) = f(\varphi(x))\det(\varphi'(x))dx_1 \wedge \cdots \wedge dx_n. \tag{3.3.2}$$

[10] *Hint:* Use the definition $\int_U f = \sum_{j \in \mathcal{J}} \int_{\mathbb{R}^n} \rho_j f$. You need to prove that $\{\rho_j \circ \varphi\}_{j \in \mathcal{J}}$ is a continuous partition of unity of V.

Example 3.3.3. Let $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$. Then

$$\varphi^*(dx) = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta$$
$$\varphi^*(dy) = d(r\sin\theta) = \sin\theta dr + r\cos\theta d\theta.$$

Wedging those together gives

$$(\cos\theta dr - r\sin\theta d\theta)\wedge(\sin\theta dr + r\cos\theta d\theta) = r\cos^2\theta dr\wedge d\theta - r\sin^2\theta d\theta\wedge dr = rdr\wedge d\theta,$$

and so $\varphi^*(dx\wedge dy) = rdr\wedge d\theta$. This lines up with the familiar formula from integration in polar coordinates (which we will prove shortly):

$$\int_V f(x,y)\,dx\,dy = \int_U f(r\cos\theta, r\sin\theta)r\,dr\,d\theta,$$

where U and V are open subsets of $\mathbb{R}^2$ such that $\varphi\colon V\to U$ is a diffeomorphism, for instance $V = (0,\infty)\times(0,2\pi)$ and $U = \{(x,y)\in\mathbb{R}^2\colon x>0 \text{ or } y\neq 0\}$.

Exercise 3.3.4.

(1) Let $\varphi(u,v) = (x(u,v), y(u,v), z(u,v)) = (uv, u+v, u^v)$. Let $\alpha = xdy - ydx$ and $\beta = dx\wedge dy + x^2 dy\wedge dz$. Find $\varphi^*\alpha$, $\varphi^*\beta$, and $\varphi^*(\alpha\wedge\beta)$, when $u>0$ and $v>0$.
(2) Let $v_j = e_1+\cdots+e_j$, where $e_1, \ldots, e_n$ are the standard basis vectors in $\mathbb{R}^n$.
   (a) For each j, k, ℓ, and m, evaluate $(dx_j\wedge dx_k)(v_\ell, v_m)$.
   (b) What are the possible values of $(dx_{j_1}\wedge\cdots\wedge dx_{j_n})(v_{k_1},\ldots,v_{k_n})$?

3.4. **Changes of variables which are eventually the identity.**

Theorem 3.4.1. *Let $f\colon\mathbb{R}^n\to\mathbb{R}$ be C^1 and compactly supported, and let $\varphi\colon\mathbb{R}^n\to\mathbb{R}^n$ be C^2. If $\varphi(x) = x$ when $|x|$ is large enough, then*

$$\int_{\mathbb{R}^n} f(\varphi(x))\det\varphi'(x)\,dx = \int_{\mathbb{R}^n} f(y)\,dy. \tag{3.4.2}$$

Proof. Let

$$g(y_1,\ldots,y_n) = \int_{-\infty}^{y_1} f(z,y_2,\ldots,y_n)\,dz, \qquad \partial_{y_1}g(y) = f(y).$$

Then use the algebra of forms to rewirte the n form corresponding to the left side of (3.4.2) as follows:

$$\begin{aligned}f(\varphi(x))\det(\varphi'(x))dx_1\wedge\cdots\wedge dx_n &= (\partial_{y_1}g)(\varphi(x))\det(\varphi'(x))\,dx_1\wedge\cdots\wedge dx_n\\ &= \varphi^*(\partial_{y_1}g(y)dy_1\wedge\cdots\wedge dy_n) = \varphi^*(dg\wedge dy_2\wedge\cdots\wedge dy_n) = \det(\tilde\varphi'(x))\,dx_1\wedge\cdots\wedge dx_n,\end{aligned}$$

where in the second equality we used (3.3.2), in the third (3.3.1), and in the fourth (3.3.2) again, with $\tilde\varphi(x) = (g\circ\varphi, \varphi_2,\ldots,\varphi_n)$. In other words,

$$f(\varphi(x))\det(\varphi'(x)) = \sum_\sigma \operatorname{sgn}(\sigma)\partial_{\sigma(1)}(g\circ\varphi)\partial_{\sigma(2)}\varphi_2\cdots\partial_{\sigma(n)}\varphi_n,$$

where the sum is over permutations of $\{1,\ldots,n\}$, the partials on the right are with respect to the x variables and everything is evaluated at x. We now plug this into the left side of (3.4.2) and integrate by parts over a cube $(-c,c)^n$ large enough to encompass the supports of f and $\varphi - I$, taking the $\partial_{\sigma(1)}$ derivative off $\partial_{\sigma(1)}(g\circ\varphi)$ and putting it on $\partial_{\sigma(2)}\varphi_2\cdots\partial_{\sigma(n)}\varphi_n$, to get

$$\int_{\mathbb{R}^n} f(\varphi(x))\det(\varphi'(x))\,dx = \quad \textit{boundary terms} \quad -$$
$$\int_{\mathbb{R}^n}\sum_\sigma \operatorname{sgn}(\sigma)(g\circ\varphi)\sum_{k=2}^n \partial_{\sigma(2)}\varphi_2\cdots\partial_{\sigma(k-1)}\varphi_{k-1}\partial_{\sigma(1)}\partial_{\sigma(k)}\varphi_k\partial_{\sigma(k+1)}\varphi_{k+1}\cdots\partial_{\sigma(n)}\varphi_n.$$

But $\partial_{\sigma(1)}\partial_{\sigma(k)}\varphi_k = \partial_{\sigma(k)}\partial_{\sigma(1)}\varphi_k$ by Clairaut's Theorem (Theorem 1.1.2). Hence, for every k, we have $\sum_\sigma \operatorname{sgn}(\sigma)\partial_{\sigma(2)}\varphi_2\cdots\partial_{\sigma(k-1)}\varphi_{k-1}\partial_{\sigma(1)}\partial_{\sigma(k)}\varphi_k\partial_{\sigma(k+1)}\varphi_{k+1}\cdots\partial_{\sigma(n)}\varphi_n = 0$. This leaves the boundary terms, and to simplify these we use the fact that $\varphi = I$ on the boundary and $g = 0$ on all boundary faces except $\{c\}\times[-c,c]^{n-1}$, giving

$$\int_{\mathbb{R}^n} f(\varphi(x))\det(\varphi'(x))\,dx = \int_{\mathbb{R}^{n-1}} g(c,y_2,\ldots,y_n)\,dy_2\cdots dy_n = \int_{\mathbb{R}^n} f(y)\,dy,$$

where we inserted the definition of g and used the fact that f is supported in $(-c,c)^n$. $\square$

3.5. **Diffeomorphism change of variables.**

THEOREM 3.5.1. *Suppose that $\varphi\colon V\to U$ is a diffeomorphism and $f\colon U\to\mathbb{R}$ is compactly supported and C^1. Then*

$$\int_V f(\varphi(x))|\det\varphi'(x)|\,dx = \int_U f(y)\,dy. \tag{3.5.2}$$

Proof. Since φ is a diffeomorphism, $\det(\varphi'(x)) \neq 0$ on V. By switching two rows or columns of φ, we can reduce the case $\det(\varphi'(x)) < 0$ to the case $\det(\varphi'(x)) > 0$, so from now on assume $\det(\varphi'(x)) > 0$ for all x in V.

Further, by a partition of unity argument, it is enough to show that every p in the support of f has a neighborhood U_p such that (3.5.2) holds with f replaced by f_p, where f_p is a C^1 function supported in U_p. By composing with a linear function and a translation we may assume that $p = 0$ and that $D\varphi(0) = I$.

By Theorem 3.4.1, and using the injectivity of φ, it is enough to prove that there exists a C^2 injective map $\Phi\colon\mathbb{R}^n\to\mathbb{R}^n$ such that $\Phi = \varphi$ near 0 and $\Phi = I$ off a compact set.

Let
$$\Phi(x) = \varphi(x)b(x/\varepsilon) + x(1 - b(x/\varepsilon)),$$
where $b\in C^2(\mathbb{R}^n)$ is 1 on $B_{1/2}$ and vanishes outside of B_1, where B_r is the open ball of radius r centered at the origin, and $\varepsilon > 0$ is to be determined. Then $\Phi = \varphi$ in $B_{\varepsilon/2}$ and $\Phi = I$ off B_ε, so it remains to choose ε small enough that Φ is injective. But

$$|x - \tilde{x}| \le |\Psi(x) - \Psi(\tilde{x})| + |\Phi(x) - \Phi(\tilde{x})|,$$

where $\Psi(x) = \Phi(x) - x = b(x/\varepsilon)(\varphi(x) - x)$. It is enough to show that $\sup_{z\in B_\varepsilon}\|\Psi'(z)\| \to 0$ as $\varepsilon\to 0$, because then for ε small enough we have $|\Psi(x) - \Psi(\tilde{x})| \le |x - \tilde{x}|/2$, and thus $|x - \tilde{x}| \le 2|\Phi(x) - \Phi(\tilde{x})|$ which implies Φ is injective.

By the product rule,

$$\|\Psi'(z)\| \le |b(z/\varepsilon)|\|\varphi'(z) - I\| + |b'(z/\varepsilon)||\varphi(z) - z|/\varepsilon.$$

To handle the first term, use $\sup_{z\in B_\varepsilon}|b(z/\varepsilon)| \le \max|b|$, and $\sup_{z\in B_\varepsilon}\|(\varphi'(z)-I)\| \to 0$ as $\varepsilon \to 0$ because $\varphi'(0)=I$ and φ' is continuous. But b stays bounded and $\varphi'(z)\to I$. To handle the second term, similarly use $|b'(z/\varepsilon)| \le \max|b'|$, and $|\varphi(z)-z|/\varepsilon \le \sup_{w\in B_\varepsilon}\|\varphi'(w)-I\| \to 0$ as $\varepsilon\to 0$. □

4. Integration on manifolds

The change of variables formula (3.5.2) gives rise to a natural way to define integration of a k-form over a k-dimensional manifold $M \subset \mathbb{R}^n$, in a way that depends on the *orientation* of the manifold. We begin with the case $k=1$.

4.1. **Integration of one-forms on curves.**

Definition 4.1.1. Let $C\subset\mathbb{R}^n$ be an oriented curve, and $\gamma\colon (a,b)\to C$ a parametrization of it. Let $\omega = \sum_{j=1}^n F_j(x)dx_j$ be a one form defined on C. Then $\gamma^*\omega = \sum_{j=1}^n F_j(\gamma(t))\gamma_j'(t)dt$, and we define

$$\int_C \omega = \int_a^b \gamma^*\omega = \int_a^b \sum_{j=1}^n F_j(\gamma(t))\gamma_j'(t)\,dt.$$

The above integral is also written as

$$\int_C F\cdot dr = \int_C F_1\,dx_1 + \cdots + F_n\,dx_n = \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt, \qquad F=(F_1,\dots,F_n).$$

It computes the work done by the force F moving a particle along the curve C.

Theorem 4.1.2. *Definition 4.1.1 is independent of parametrization.*

Proof. This is where the change of variables formula comes in. Let $\sigma\colon (c,d)\to C$ be another parametrization of C with the same orientation. Putting $t=(\gamma^{-1}\circ\sigma)(s)$ yields

$$\int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt = \int_c^d F(\sigma(s))\cdot\gamma'((\gamma^{-1}\circ\sigma)(s))(\gamma^{-1}\circ\sigma)'(s)\,ds = \int_c^d F(\sigma(s))\sigma'(s)\,ds.$$

□

The fundamental theorem of calculus makes the link with exterior differentiation:

$$\int_C df = \int_a^b \sum_{j=1}^n \partial_j f(\gamma(t))\gamma_j'(t)dt = \int_a^b \frac{d}{dt}f(\gamma(t))\,dt = f(\gamma(b)) - f(\gamma(a)).$$

We write this as

$$\int_C df = \int_{\partial C} f, \tag{4.1.3}$$

where C is an oriented curve from $\gamma(a)$ to $\gamma(b)$, and ∂C is the boundary of C with the corresponding orientation, namely the two point set $\{\gamma(a),\gamma(b)\}$ with the first point oriented by $-$ and the second by $+$. We will generalize this formula to integrals of any dimension.

Exercise 4.1.4.

(1) Evaluate

$$\int_C \sin z dx + \cos(\sqrt{y})dy + x^3dz,$$

where C is the line segment from $(1,0,0)$ to $(0,0,3)$.

(2) Let $\alpha = yzdx + xzdy + xydz$.
  (a) Find a function f such that $df = \alpha$.
  (b) Evaluate $\int_C \alpha$, where C is the parametric curve $(\cos t, e^t, \ln t)$, $1 \le t \le 2$.

4.2. **Integration of forms on manifolds.** The next simplest case is integrating n-forms over open sets in $\mathbb{R}^n$.

DEFINITION 4.2.1. Given $\omega = f(x)dx_1 \wedge \cdots \wedge dx_n$ defined on an open set $U \subset \mathbb{R}^n$, define the integral of ω over U with respect to the *standard orientation* by

$$\int_U \omega = \int_U f,$$

and define the integral of ω over U with respect to the *reverse orientation* by

$$\int_U \omega = -\int_U f.$$

If we write instead $\omega = g(x)dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(n)}$ for some permutation σ, then $\int_U g = \int_U \omega$ with respect to the standard orientation if $\operatorname{sgn}(\sigma) = 1$ and with respect to the reverse orientation if $\operatorname{sgn}(\sigma) = -1$. When $n = 1$, the standard orientation corresponds to $\int_{-\infty}^{\infty}$ and the reverse orientation to $\int_{\infty}^{-\infty}$.

We now treat the case of general k-forms.

DEFINITION 4.2.2. Let $U \subset \mathbb{R}^k$ be open, let $\varphi\colon U \to M$ be a coordinate chart, and let ω be a k-form defined on M, i.e. ω is a sum of terms of the form $f(x)dx_{i_1} \wedge \cdots \wedge dx_{i_k}$, with each f a smooth function defined on M. Then define *the integral of ω over $\varphi(U)$ with respect to the orientation induced by φ* by

$$\int_{\varphi(U)} \omega = \int_U \varphi^* \omega,$$

where for the integration over U we use the standard orientation. We say that two coordinate charts $\varphi\colon U \to M$ and $\psi\colon V \to M$ with $\varphi(U) = \psi(V)$ *induce the same orientation* if $\det((\varphi^{-1} \circ \psi)'(y)) > 0$ for all $y \in V$.

THEOREM 4.2.3. *If $\varphi\colon U \to M$ and $\psi\colon V \to M$ with $\varphi(U) = \psi(V)$ induce the same orientation then*

$$\int_U \varphi^* \omega = \int_V \psi^* \omega.$$

Proof. Write $\varphi^*\omega = f(x)\, dx_1 \wedge \cdots dx_n$ and $\psi^*\omega = g(y)\, dy_1 \wedge \cdots dy_n$. Then

$$\psi^*\omega = (\varphi^{-1} \circ \psi)^* \varphi^* \omega = f(\varphi^{-1}(\psi(y))) \det((\varphi^{-1} \circ \psi)'(y))\, dy_1 \wedge \cdots dy_n,$$

and hence $g(y) = f(\varphi^{-1}(\psi(y))) \det((\varphi^{-1} \circ \psi)'(y))$. Thus the Theorem states that

$$\int_U f = \int_V f(\varphi^{-1}(\psi(y))) \det((\varphi^{-1} \circ \psi)'(y))\, dy,$$

and this follows from the change of variables formula (3.5.2). □

EXERCISE 4.2.4. Let (w, x, y, z) be the standard coordinates on $\mathbb{R}^4$. Let

$$\alpha = dw \wedge dx + dy \wedge dz, \qquad \beta = dw \wedge dy + dx \wedge dz.$$

Let a and b be given positive numbers, and let M be the surface given by

$$w^2 + x^2 = a^2, \qquad y^2 + z^2 = b^2, \quad w > 0, \quad x > 0, \quad y > 0, \quad z > 0.$$

(1) Simplify $\alpha \wedge \alpha$ and $\alpha \wedge \beta$.
(2) Evaluate $\int_M \alpha$ and $\int_M \beta$, where M is oriented so that $\int_M \beta > 0$.

To define more general integration of forms on manifolds, we use a partition of unity. More specifically, let M be a manifold, and let $\{\rho_j\}_{j\in\mathcal{J}}$ be a continuous partition of unity on M such that each ρ_j is supported in a coordinate patch $\varphi_j(U_j)$. Then define

$$\int_M \omega = \sum_{j\in\mathcal{J}} \int_M \rho_j \omega = \sum_{j\in\mathcal{J}} \int_{U_j} \varphi_j^*(\rho_j\omega). \tag{4.2.5}$$

In most computations one finds a way to avoid dealing with the partition of unity. In the example below, a single coordinate chart covers the whole manifold except for a negligible set.

Example 4.2.6. Let $\omega = z^2 dx \wedge dy + \cos(x) dx \wedge dz$, and let M be given by $0 < z = \sqrt{x^2+y^2} < 1$. Let $\varphi(r,\theta) = (r\cos\theta, r\sin\theta, r)$, and then

$$\varphi^*\omega = r^3 dr \wedge d\theta + \cos(r\cos\theta)(r\sin\theta) dr \wedge d\theta.$$

Thus, with respect to the orientation induced by φ,

$$\int_M \omega = \int_{-\pi}^{\pi} \int_0^1 (r^3 + \cos(r\cos\theta)(r\sin\theta))\, dr\, d\theta = \frac{\pi}{2}.$$

We get the same answer if $\cos(x)$ is replaced by $f(x)$ where f is any continuous function, because $f(r\cos\theta)(r\sin\theta) = \frac{d}{d\theta}F(r\cos\theta)$ integrates to 0 from $-\pi$ to π.

Two remarks on the above example:

(1) There is a natural correspondence[11] between vector fields in $\mathbb{R}^3$ and 2 forms, given by

$$(F_1, F_2, F_3) \leftrightarrow F_1\, dy \wedge dz - F_2\, dx \wedge dz + F_3\, dx \wedge dy.$$

With that correspondence, $\int_M \omega$ gives the *flux* of the vector field. For more on this, see Section 4.5. Note that in the above example $(0, -f(x), 0)$ has no net flux through M because the flux through any (x,y,z) in M is precisely balanced by that through $(x,-y,z)$.

(2) In the above calculation we treated as negligible the set where $\theta = \pm\pi$. To be more precise about this, prove that

$$\int_M \omega = \lim_{\varepsilon\to 0^+} \int_{-\pi+\varepsilon}^{\pi-\varepsilon} \int_0^1 (r^3 + \cos(r\cos\theta)(r\sin\theta))\, dr\, d\theta,$$

by using a partition of unity consisting of two functions $\{\rho_1, \rho_2\}$, such that $\rho_1 = 1$ on the set where $-\pi + \varepsilon \le \theta \le \pi - \varepsilon$.

4.3. **Stokes' Theorem.**

The general Stokes' Theorem says that

$$\int_M d\omega = \int_{\partial M} \omega, \tag{4.3.1}$$

where M is a suitable manifold of dimension n, ω is an $n-1$ form on M, and ∂M is the boundary of M. The one-dimensional case, where M is a curve, is in (4.1.3).

[11] The higher dimensional version of this is

$$(F_1, \dots, F_n) \leftrightarrow F_1\, dx_2 \wedge \cdots \wedge dx_n - F_2\, dx_1 \wedge dx_3 \wedge \cdots \wedge dx_n + \cdots + (-1)^{n-1} F_n\, dx_1 \wedge \cdots \wedge dx_{n-1}.$$

It is based on the *Hodge star* operation, which you can read more about in Exercises 10–15 of Chapter 1 of do Carmo.

To get a feel for the higher-dimesional versions, we begin with the case where M is a rectangle in $\mathbb{R}^2$. Let a and b be positive numbers, and let C_1 be the segment from $(0,0)$ to $(a,0)$, C_2 the segment from $(a,0)$ to (a,b), C_3 the segment from (a,b) to $(0,b)$ and C_4 the segment from $(0,b)$ to $(0,0)$. Let C be the concatenation of C_1, C_2, C_3, C_4, so the whole perimeter of the rectangle, and let R be the interior of the rectangle. Then

$$\int_C F_1 dx_1 = \int_{C_1} F_1 dx_1 + \int_{C_3} F_1 dx_1 = \int_0^a (F_1(t,0) - F_1(t,b))dt = -\int_0^a \int_0^b \partial_2 F_1(t,s) ds dt = -\int_R \partial_2 F_1.$$

Similarly,

$$\int_C F_2 dx_2 = \int_R \partial_1 F_2,$$

and adding these gives *Green's formula* for a rectangle:

$$\int_R (\partial_1 F_2 - \partial_2 F_1) = \int_C F_1 dx_1 + F_2 dx_2, \tag{4.3.2}$$

which is a special case of (4.3.1).

For more general higher dimensional cases, we use a partition of unity and convenient coordinates to simplify the calculation. The fundamental case is the following.

THEOREM 4.3.3. *Let ω be a compactly supported $n-1$ form on $\mathbb{R}^n$. Let $H = \{x \in \mathbb{R}^n : x_1 < 0\}$. Then*

$$\int_H d\omega = \int_{\partial H} \omega,$$

with H and ∂H both carrying the standard orientations, induced respectively by $dx_1 \wedge \cdots dx_n$ and $dx_2 \wedge \cdots \wedge dx_n$.

Proof. By linearity it is enough to consider the case $\omega = a(x) dx_1 \wedge \cdots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \cdots \wedge dx_n$, where $a(x)$ is compactly supported. Then $d\omega = (-1)^{k-1} \partial_{x_k} a(x) dx_1 \wedge \cdots \wedge dx_n$. We have two cases.

(1) If $k = 1$, then

$$\int_H d\omega = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \partial_{x_1} a(x)\, dx_1 \cdots dx_n = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} a(x)\, dx_2 \cdots dx_n = \int_{\partial H} \omega.$$

(2) If $k \neq 1$, then we do the same calculation but with the innermost integral being the one with respect to x_k. Then $\int_H d\omega = 0$ because the innermost integral is

$$\int_{-\infty}^{\infty} \partial_{x_k} a(x) dx_k = 0,$$

due to the compact support of $a(x)$. Meanwhile $\int_{\partial H} \omega = 0$ because $x_1 = 0$ on H and hence $dx_1 = 0$ there as well.

$\square$

Stokes' theorem for more general manifolds with boundary can be deduced from Theorem 4.3.3 by partition of unity and coordinate charts.

$$\begin{aligned}\int_M d\omega &= \sum_{j\in\mathcal{J}} \int_M d(\rho_j\omega) = \sum_{j\in\mathcal{J}} \int_{U_j} \varphi_j^*(d(\rho_j\omega)) = \sum_{j\in\mathcal{J}} \int_{U_j} d(\varphi_j^*(\rho_j\omega)) \\ &= \sum_{j\in\mathcal{J}} \int_{\partial U_j} \varphi_j^*(\rho_j\omega) = \sum_{j\in\mathcal{J}} \int_{\partial M} \rho_j\omega = \int_M \omega.\end{aligned}$$

One can also consider manifolds with corners, such as polygons and polyhedra, by repeating Theorem 4.3.3 with H replaced by $(-\infty,0)^n$. More general domains, such as manifolds with cusps, can be treated by decomposition: see Section 4.3 of Taylor.

4.4. **Volume.** Our next step is to compute volume of manifolds, also known as length and surface area in the 1- and 2- dimensional cases. We begin with parallelipipeds.

Theorem 4.4.1. *The volume of the k-dimensional parallelipiped in $\mathbb{R}^n$ spanned by the vectors $v_1,\dots,v_k$ is $\sqrt{\det A^TA}$, where A is the matrix whose columns are $v_1,\dots,v_k$.*

Proof. We begin with the case $k=n$. Then the volume is $|\det A|$ as in Section 3.2, and

$$\sqrt{\det A^TA} = \sqrt{\det A^T \det A} = \sqrt{\det A \det A} = |\det A|.$$

Next, if $k<n$, we reduce to the above case in the following way. Take an orthogonal matrix R such that the columns of $B=AR$ all lie in the subspace $x_{k+1}=\cdots=x_n=0$. Call these columns $w_1,\dots,w_k$, and note that they span a congruent parallelipiped. Its volume is $\sqrt{\det B^TB}$ by the above case, and

$$\det B^TB = \det R^TA^TAR = \det R^T \det A^TA \det R = \det A^TA,$$

because the determinant of an orthogonal matrix is ±1. □

This leads us to the following definition:

Definition 4.4.2. Let $U\subset\mathbb{R}^k$ be open, let $\varphi\colon U\to M$ be a coordinate chart, and let f be a continuous function on $\varphi(U)$. The *the surface integral of f over $\varphi(U)$* is defined by

$$\int_{\varphi(U)} f\,dS = \int_U f(\varphi(x))\sqrt{\det \varphi'(x)^T\varphi'(x)}\,dx.$$

We can then define $\int_M f\,dS$ using a partition of unity as in (4.2.5). If $f=1$, then this is the *volume* of M.[12]

Example 4.4.3.

(1) If $k=1$, and $U=(a,b)$, and we denote $\varphi=\gamma$, then we have

$$\int_{\gamma(a,b)} f\,dS = \int_a^b f(\gamma(t))|\gamma'(t)|\,dt,$$

which is the usual arclength integral.

[12]The differential form given in coordinates by $\sqrt{\det\varphi'(x)^T\varphi'(x)}\,dx_1\wedge\cdots\wedge dx_k$ is called the *volume form.* It is discussed in Section 8.5.1 of Shifrin.

(2) If $k = n$, then we have

$$\int_{\varphi(U)} f\,dS = \int_U f(\varphi(x))|\det \varphi'(x)|\,dx = \int_{\varphi(U)} f,$$

by the change of variables formula (3.5.2). Thus, in this case the manifold is just an open set of $\mathbb{R}^n$ and the surface integral reduces to the ordinary integral.

(3) If M is the graph of a function $h\colon U \to \mathbb{R}$, where $U \subset \mathbb{R}^k$, then $\varphi(x) = (x, h(x))$, and $\varphi'(x)$ is a block matrix, consisting of a $k \times k$ identity matrix on top of $h'(x)$. Then $\varphi'(x)^T\varphi'(x)v = |h'(x)|^2 v$ if v is a multiple of $h'(x)$ and $\varphi'(x)^T\varphi'(x)v = 1$ if v is perpendicular to $h'(x)$, implying that $\det \varphi'(x)^T\varphi'(x) = 1 + |h'(x)|^2$ and thus

$$\int_M f\,dS = \int_U f(x, h(x))\sqrt{1 + |h'(x)|^2}\,dx. \tag{4.4.4}$$

(4) For the surface discussed in Example 4.2.6, we have

$$\varphi'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 1 & 0 \end{pmatrix}, \qquad \det \varphi'(r,\theta)^T\varphi'(r,\theta) = \det \begin{pmatrix} 2 & 0 \\ 0 & r^2 \end{pmatrix} = 2r^2,$$

and so the area is

$$\int_M dS = \int_0^1 \int_{-\pi}^{\pi} \sqrt{2}r\,dr\,d\theta = \sqrt{2}\pi r^2.$$

4.5. **Flux.** The integral of a $n-1$-form over a $n-1$ dimensional manifold computes flux of a vector field. More specifically,

$$\int_M F\cdot\nu\,dS = \int_M \omega, \tag{4.5.1}$$

where ν is a unit normal vector to M, and the vector field F and $n-1$ form ω are related as in Remark (1) on Example 4.2.6. To prove (4.5.1), it is convenient to work in coordinates as in (4.4.4), where $k = n-1$, $(x, y) = (x_1, \ldots, x_k, y)$, $\varphi(x) = (x, h(x))$, and $M = \{(x, y)\colon x \in U,\ y = h(x)\}$. We write $F = (F_1, \ldots, F_k, G)$, and[13] $\nu(x) = \pm(-h'(x), 1)/\sqrt{1 + |h'(x)|^2}$, so that

$$\int_M F\cdot\nu\,dS = \pm\int_U \Big(G(x, h(x)) - \sum_{j=1}^k F_j(x, h(x))\partial_j h(x)\Big)\,dx.$$

Meanwhile

$$\omega = (-1)^k G\,dx_1\wedge\cdots\wedge dx_k + \sum_{j=1}^k (-1)^{j-1} F_j\,dx_1\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_k\wedge dy,$$

so that

$$\begin{aligned}\varphi^*\omega &= (-1)^k\varphi^* G\,dx_1\wedge\cdots\wedge dx_k + \sum_{j=1}^k (-1)^{j-1}\varphi^* F_j\,dx_1\wedge\cdots\wedge dx_{j-1}\wedge dx_{j+1}\wedge\cdots\wedge dx_k\wedge dh \\ &= (-1)^k\Big(G(x, h(x)) - \sum_{j=1}^k F_j(x, h(x))\partial_j h(x)\Big)\,dx_1\wedge\cdots\wedge dx_k.\end{aligned}$$

[13] To compute ν, use the fact that if $f(x, y) = h(x) - y$, then $M = f^{-1}(0)$ so $f'(x) = (h'(x), -1)$ is a normal vector to the surface at $(x, h(x))$ because the gradient of a function is perpendicular to its level sets.

Thus (4.5.1) holds, provided the orientation of ν on the left side of the equation is compatible with that of M on the right side of the equation. More specifically, the orientation induced by φ on the right side corresponds to the upward normal $\nu(x) = (-h'(x), 1)/\sqrt{1 + |h'(x)|^2}$ when k is even and to the downward normal $\nu(x) = (h'(x), -1)/\sqrt{1 + |h'(x)|^2}$ when k is odd.