MA 442 first midterm review problems

Hopefully final version as of February 7th.

The first midterm will be in class on Monday, February 15th. No notes or electronic devices allowed. Most of the problems on the exam will be closely based on ones from the list below. For each problem, you must justify your answers. Please let me know if you have a question or find a mistake.

1. For which x and y in $\mathbb{R}^n$ does $|x+y| = |x| + |y|$? Give an algebraic proof.
2. Let $A\colon \mathbb{R}^m \to \mathbb{R}^n$ be linear. Show that

$$\sup\{|Ax|\colon |x| = 1\} = \inf\{C\colon |Ax| \le C|x| \text{ for all } x\}$$

3. Recall the definition of the derivative of $f\colon \mathbb{R}^m \to \mathbb{R}^n$, and the statement of the chain rule for the derivative of its composition with $g\colon \mathbb{R}^n \to \mathbb{R}^p$. Use these, together with the single variable calculus fact that if $h(t) = t^\lambda$ for some real λ, then $h'(t) = \lambda t^{\lambda-1}$, to show that $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = |x|^\alpha$ is differentiable for any real α at any $a \in \mathbb{R}^n \setminus 0$ and find its derivative. For which values of α is it differentiable at 0?
4. Let $g\colon \mathbb{R}^m \to \mathbb{R}^n$, and suppose there are a constant $\lambda < 1$ and an open set $U \in \mathbb{R}^n$ such that $|g(x) - g(y) - (x-y)| \le \lambda |g(x) - g(y)|$ for all x and y in U.
   (a) Prove that g is continuous at all points of U.
   (b) Find a number c depending on λ, as small as you can, such that if g is differentiable, then $|Dg(x)h| \le c|h|$ for all $x \in U$ and $h \in \mathbb{R}^m$
5. Let $M(n, \mathbb{R})$ denote the space of real $n \times n$ matrices.
   (a) Define $E\colon M(n,\mathbb{R}) \to M(n,\mathbb{R})$ by

$$E(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}.$$

   Show that E is differentiable at 0 and that $DE(0)Y = Y$.

   (b) Use the series for $(1+x)^{-1}$ to find coefficients a_n such that $\ln(1+x) = \sum_{n=0}^{\infty} a_n x^n$ for x small enough. Define $L\colon U \to M(n,\mathbb{R})$ by

$$L(A) = \sum_{n=0}^{\infty} a_n A^n,$$

   where U is a neighborhood of 0 in $M(n,\mathbb{R})$ such that the sum converges. Show that L is differentiable at 0 and that $DL(0)Y = Y$.
6. Spivak problem 2-12.