MA 442 second midterm review problems

Hopefully final version as of March 29th

The second midterm will be in class on April 5th. No notes or electronic devices are allowed. The problems on the exam will be closely based on parts of problems from the list below. For each problem, you must justify your answers. Please let me know if you have a question or find a mistake.

1. Let a, b, c, d be the side lengths of a quadrilateral, α be the interior angle between a and b, β the interior angle between b and c, γ the interior angle between c and d. By the law of cosines applied at opposite angles, and the fact that the exterior angles add to 2π, we have

$$a^2 + b^2 - 2ab\cos\alpha = c^2 + d^2 - 2cd\cos\gamma,$$
$$b^2 + c^2 - 2bc\cos\beta = a^2 + d^2 - 2ad\cos(\alpha+\beta+\gamma).$$

Given a, b, c, $d > 0$, let A be the set of $(\alpha, \beta, \gamma) \in (0, \pi)^3$ which solve this system. Use the implicit function theorem to show that this system determines (β, γ) as a C^∞ function of α near any point of A such that $\alpha+\beta+\gamma > \pi$. Find $\beta'(\alpha)$ and $\gamma'(\alpha)$ when $a = 2$, $b = c = d = 1$, $\alpha = \pi/3$, and $\beta = \gamma = 2\pi/3$.

2. Let a, b, c, d, be given positive numbers, and let f, g be given C^1 functions from $\mathbb{R}^2$ to $\mathbb{R}$. Consider the system of differential equations

$$\dot{x}_1 = ax_1(b - x_1) + \varepsilon f(x_1, x_2), \quad x_1(0, \varepsilon) = \xi_1,$$
$$\dot{x}_2 = cx_2(d - x_2) + \varepsilon g(x_1, x_2), \quad x_2(0, \varepsilon) = \xi_2,$$

where ε is a small parameter, and $\xi_1 \geq 0$ and $\xi_2 \geq 0$ are initial conditions. Here x_1 and x_2 are populations of two species.

   (a) Find the unique $\xi_1^* > 0$ and $\xi_2^* > 0$ such that the system is solved by $x_j(t, 0) = \xi_j^*$ for $j = 1$ and 2, and for all t. (This is the coexistence equilibrium without interaction.)

   (b) Use the implicit function theorem to show that there exist $\varepsilon_0 > 0$ and C^1 functions $\varphi_1\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ and $\varphi_2\colon (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}$ such that the system is solved by $\varphi_j(0) = \xi_j^*$ and $x_j(t, \varepsilon) = \varphi_j(\varepsilon)$ for $j = 1$ and 2, and for all t. (Adding a small interaction perturbs the equilibrium slightly.)

   (c) Find $\varphi_1'(0)$ and $\varphi_2'(0)$ in terms of a, b, c, d, f, g. (This gives the first order change in the equilibrium).

3. Let $A \subset \mathbb{R}^2$ be an open rectangle, let u and v be C^∞ functions $A \to (0, \infty)$, and let

$$G(\alpha, \beta) = \begin{pmatrix} u(\alpha, \beta) & 0 \\ 0 & v(\alpha, \beta) \end{pmatrix}.$$

   (a) Assume there is a c such that $u(\alpha, \beta) \geq u(\alpha, c)$ for all α and β in A. For any C^∞ parametrized curve $\gamma\colon (0, 1) \to A$, define the G-length of γ by $\int_0^1 \sqrt{[G(\gamma(t))\dot{\gamma}(t)] \cdot \dot{\gamma}(t)}dt$. Show that, for any (a, c) and (b, c) in A, the G-length of any parametrized curve from (a, c) to (b, c) is $\geq$ the G-length of a line segment from (a, c) to (b, c), with equality if and only if γ monotonically parametrizes the line segement.

(b) The result of part (a) concerns horizontal line segments: state and prove an analogous result about vertical line segments, suitably modifying the assumption in the first sentence of part (a).

(c) Which line segments do the results of part (a) and (b) apply to for the examples of the helicoid, sphere, torus, ellipsoid, and cylinder? (On the actual exam you would be given the relevant parametrization of the surface as in equation (6) and Exercises 2.6 and 2.9 of `https://www.math.purdue.edu/~kdatchev/442/geodesics.pdf`.)

4. Given that the volume of a ball of radius R in $\mathbb{R}^3$ is $4\pi R^3/3$, use the change of variables formula to find the volume of the ellipsoid

$$a^2x^2 + b^2y^2 + c^2z^2 < d^2,$$

where a, b, c, and d are given nonzero real numbers.

5. (a) Use the fact that if f is continuous then $\int_a^b \int_c^d f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dydx$ for any real a, b, c, and d to show that if F is C^1 then $\frac{d}{dx}\int_a^b F(x,y)dy = \int_a^b \partial_x F(x,y)dy$.

(b) Let

$$f(t) = \left(\int_0^t e^{-x^2}dx\right)^2, \qquad g(t) = \int_0^1 \frac{e^{-t^2(1+x^2)}}{1+x^2}dx.$$

Show that $f'(t) + g'(t) = 0$ for all t, hence that $f(t) + g(t) = \pi/4$ for all t, and hence that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$.