MA 442 midterm review problems

Version as of February 22nd.

The midterm will be in class on Tuesday, February 27th. No notes, books, or electronic devices will be allowed. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Justify your answers. Please let me know if you have a question or find a mistake.

1. Let $f\colon \mathbb{R}^{2n} \to \mathbb{R}$ be given by $f(x) = |x|^2$. Find $f'(a)h$ when $a = (1, 2, \ldots, 2n)$ and $h = (1, -1, 1, -1, \ldots, -1)$.

2. Let $A\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. Prove that
$$\sup\{|Av|\colon |v| = 1\} = \sup\{|Av|/|v|\colon v \neq 0\} = \sup\{|Av|\colon |v| \leq 1\}.$$
Also prove that, for any $c > 0$, the above are all equal to $\sup\{|Av/c|\colon |v| = c\}$

3. Let $U \subset \mathbb{R}^n$ be open, let $g\colon U \to \mathbb{R}^n$, and suppose there is a constant $\lambda < 1$ such that $|g(x) - g(y) - (x - y)| \leq \lambda |g(x) - g(y)|$ for all x and y in U.
   (a) Prove that g is continuous.
   (b) Find a number C depending on λ, as small as you can, such that if g is differentiable at x, then the operator norm of $g'(x)$ is at most C.

4. Let $M(n, \mathbb{R})$ denote the space of real $n \times n$ matrices. Use the series for $(1-x)^{-1}$ to find coefficients a_n such that $\ln(1 - x) = \sum_{n=0}^{\infty} a_n x^n$ for x small enough. Define $L\colon U \to M(n, \mathbb{R})$ by
$$L(A) = \sum_{n=0}^{\infty} a_n A^n,$$
where U is a neighborhood of 0 in $M(n, \mathbb{R})$ such that the sum converges. Show that L is differentiable at 0 and that $L'(0)Y = -Y$.

5. Consider the system of equations
$$\begin{aligned} xyz &= \alpha \\ x + y + z &= \beta \\ x^{y^z} &= \gamma. \end{aligned}$$
Given $a > 2$, find the solution of the system when $(\alpha, \beta, \gamma) = (0, a, 2)$. Show that, for (α, β, γ) sufficiently near $(0, a, 2)$, the system defines (x, y, z) uniquely as a function of (α, β, γ). (Take advantage of the partial derivatives that are zero to simplify the calculation.) Find $\partial_\alpha z$, $\partial_\beta z$, $\partial_\gamma z$, $\partial_\beta x$, $\partial_\gamma x$, $\partial_\beta y$, and $\partial_\gamma y$ at $(0, a, 2)$. (Note that these are easier than $\partial_\alpha x$ and $\partial_\alpha y$.)

6. Let M consist of the points in $\mathbb{R}^4$ which are unit distance from the origin and equidistant from the 'plane' $x_2 = 1$ and the x_1 axis. Prove that M is a manifold. What is its dimension?

7. Let M be the cylinder $x^2 + y^2 = R^2$ in $\mathbb{R}^3$. Find coordinates on M in which the metric tensor is the Euclidean one (i.e. the matrix G is the identity matrix). Find the lengths of the three shortest geodesics on M from $(x, y, z) = (R, 0, 0)$ to $(x, y, z) = (0, R, 1)$. (Hint: Two of them have lengths less than $\sqrt{4\pi^2 R^2 + 1}$) Show that, for any real number N, there are geodesics on M from $(x, y, z) = (R, 0, 0)$ to $(x, y, z) = (0, R, 1)$ of length greater than N and find the length of one such geodesic.