The fundamental theorem of differential equations

Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous, let $p \in \mathbb{R}^n$, and consider the initial value problem

$$x'(t) = f(x(t)), \qquad x(0) = p. \tag{1}$$

More general initial value problems for ordinary differential equations can be reduced to this one by a change of variables. For example, solving $y'' + g(y) = 0$ with $y(0) = a$ and $y'(0) = b$ is equivalent to solving (1) with $f(x_1, x_2) = (x_2, -g(x_1))$, $p = (a, b)$, by the substitution $x(t) = (y(t), y'(t))$. The following exercise generalizes this.

Exercise 1. Given a continuous $F\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, and $(y_0, y_1, r_0) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, find f and p such that $x(t) = (y'(t + r_0), y(t + r_0), t + r_0)$ solves (1) if and only if

$$y''(r) = F(y'(r), y(r), r), \qquad y(r_0) = y_0, \quad y'(r_0) = y_1.$$

We solve (1) by converting to an integral equation. Let $T > 0$ be given. If a differentiable function $x\colon (-T, T) \to \mathbb{R}^n$ solves (1), then it solves the integral equation

$$x(t) = p + \int_0^t f(x(s))ds. \tag{2}$$

Conversely, any solution to (2) is continuous (because $t \mapsto \int_0^t g$ is continuous for any g) and hence also differentiable (by the fundamental theorem of calculus) and also solves (1).

Theorem 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^1 near some $p \in \mathbb{R}^n$. There is $T > 0$ such that there is a unique solution $x\colon [-T, T] \to \mathbb{R}^n$ to*

$$x(t) = p + \int_0^t f(x(s))ds. \tag{3}$$

To solve (1), we use the notation $B_r(p) = \{x \in \mathbb{R}^n \mid |x - p| < r\}$. Recall that if $f\colon B_r(p) \to \mathbb{R}^n$ is C^1 with $\|f'\| \le L$ for some constant L, then

$$|f(x) - f(y)| \le L|x - y|, \tag{4}$$

for all x and y in $B_r(p)$. Inequality (4) is called a Lipschitz condition and it can be proved by setting $c(t) = tx + (1 - t)y$ and writing

$$|f(x) - f(y)| = \Big|\int_0^1 \frac{d}{dt} f(c(t))dt\Big| = \Big|\int_0^1 f'(c(t))(x - y)dt\Big| \le \int_0^1 \|f'(c(t))\||x - y|dt \le L|x - y|.$$

Proof of Theorem 1. Define a recursive sequence, called a sequence of Picard iterates,

$$x_0(t) \equiv p, \qquad x_{k+1}(t) = p + \int_0^t f(x_k(s))ds. \tag{5}$$

Kiril Datchev, March 8, 2024. These are notes are under development, and questions, comments, and corrections are gratefully received at kdatchev@purdue.edu.

Take positive numbers δ, A, and L such that $|f| \le A$ and $\|f'\| \le L$ on $B_\delta(\xi)$. We require $T \le \delta/2A$ so that

$$|x_1(t) - x_0(t)| \le \int_0^t |f(p)|ds \le TA \le \delta/2.$$

We next require $T \le 1/2L$ so that, by (4),

$$|x_2(t) - x_1(t)| \le \int_0^t |f(x_1(s)) - f(x_0(s))|ds \le TL \max_{s\in[-T,T]} |x_1(s) - x_0(s)| \le \delta/4,$$

and, more generally,

$$|x_{k+1}(t) - x_k(t)| \le \int_0^t |f(x_k(s)) - f(x_{k-1}(s))|ds \le \max_{s\in[-T,T]} |x_k(s) - x_{k-1}(s)|/2 \le \cdots \le \delta 2^{-k-1}.$$

Thus, for $m > k$,

$$|x_m(t) - x_k(t)| \le \sum_{j=k}^{m-1} |x_{j+1}(t) - x_j(t)| \le \delta \sum_{j=k}^{m-1} 2^{-j},$$

which tends to 0 as $k \to \infty$, so x_k converges uniformly. Let $x(t) = \lim_{k\to\infty} x_k(t)$. Since

$$|f(x(s)) - f(x_k(s))| \le L|x(s) - x_k(s)|,$$

we can take the limit as $k \to \infty$ of (5) to get (3).

For uniqueness, see the exercise below. □

Exercise 2. Prove that if $x\colon [-T,T] \to \mathbb{R}^n$ and $\tilde{x}\colon [-T,T] \to \mathbb{R}^n$ both solve (3), and $|f(x(t)) - f(\tilde{x}(t))| \le L|x(t) - \tilde{x}(t)|$ for all $t \in [-T,T]$, and if $TL \le 1/2$, then $x(t) = \tilde{x}(t)$ for all $t \in [-T,T]$.

The above proof gives us more, namely the following stronger but more complicated result.

Theorem 2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^1 near some $q \in \mathbb{R}^n$. Suppose there are positive numbers δ, A, and L such that $|f| \le A$ and $\|f'\| \le L$ on $B_{2\delta}(q)$. Then, for any $T > 0$ such that $T \le \delta/2A$ and $T \le 1/2L$, there is a unique solution $x\colon [-T,T] \times B_\delta(q) \to \mathbb{R}^n$ to*

$$x(t,p) = p + \int_0^t f(x(s,p))ds. \tag{6}$$

Moreover the Picard iterates

$$x_0(t,p) \equiv p, \qquad x_{k+1}(t,p) = p + \int_0^t f(x_k(s,p))ds, \tag{7}$$

converge uniformly to it.

The uniform convergence tells us that x is continuous in p; this is called *continuous dependence on the initial condition.* Thus, for every $\varepsilon > 0$ there is $\delta > 0$ such if we that know the initial condition p up to accuracy δ then we know $x(t)$ up to accuracy ε for all $t \in [-T,T]$. If f has additional smoothness, then so does x, allowing us to relate ε and δ:

Theorem 3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^2 near some $q \in \mathbb{R}^n$. There there are positive numbers δ and T such that there is a unique solution $x\colon [-T,T] \times B_\delta(q) \to \mathbb{R}^n$ to (6) and this solution is C^1.*

Proof. We have already shown that there is a unique solution if r and T are small enough, and $\partial_t x$ is continuous by (6). To show that $\partial_{p_j} x$ is continuous for any j, observe that differentiating (7) gives

$$\partial_{p_j} x_0(t,p) = e_j, \qquad \partial_{p_j} x_{k+1}(t,p) = e_j + \int_0^t f'(x_k(s,p))\partial_{p_j} x_k(s,p)ds,$$

where e_j is the unit vector in the jth coordinate direction. Accordingly define

$$y_0(t,p) = e_j, \qquad y_{k+1}(t,p) = e_j + \int_0^t f'(x_k(s,p))y_k(s,p)ds.$$

Thus $y_k = \partial_{p_j} x_k$. It is enough to show that y_k converges uniformly; then we will have

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \partial_{p_j} x_k = \partial_{p_j} \lim_{k\to\infty} x_k = \partial_{p_j} x.$$

Put $z_k = (x_k, y_k)$. Then

$$z_0(t,p) = Z, \qquad z_{k+1}(t,p) = Z + \int_0^t g(z_k(s,p))ds, \tag{8}$$

with $\zeta = (p, e_j)$ and $g(a,b) = (f(a), f'(a)b)$. We now apply Theorem 2 with f replaced by g, and with p replaced by (p, e_j), to conclude that z_k converges uniformly, and so in particular y_k does. $\square$

Iterating the above, we see that if f is C^{K+1}, then in (8) g is C^K, so that $\partial_{p_j} x$ is C^{K-1}, and x is C^K. In particular, if f is C^∞, then so is x.

Example. Let $f(x) = Ax$. Then

$$\begin{aligned}
x_0(t) &= p, \\
x_1(t) &= p + \int_0^t Apds = (1 + At)p, \\
x_2(t) &= p + \int_0^t (1 + As)pds = (1 + At + \tfrac{1}{2}At^2)p, \\
&\dots \\
x_k(t) &= \sum_{k=0}^n \frac{(At)^k}{k!} p, \\
y_k(t) &= \sum_{k=0}^n \frac{(At)^k}{k!} e_j, \\
z_k(t) &= \Big(\sum_{k=0}^n \frac{(At)^k}{k!} p, \sum_{k=0}^n \frac{(At)^k}{k!} e_j \Big), \qquad g(a,b) = (Aa, Ab).
\end{aligned}$$