EXISTENCE AND UNIQUENESS OF SOLUTIONS TO ORDINARY DIFFERENTIAL EQUATIONS

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Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be continuous, let $\xi \in \mathbb{R}^n$, and consider the initial value problem

$$\dot{x}(t) = f(x(t)), \qquad x(0) = \xi. \tag{1}$$

Here and below $\dot{x}$ means the derivative with respect to t of x. Let I be an interval containing 0. A differentiable function $x\colon I \to \mathbb{R}^n$ solves (1) if and only if it solves the integral equation

$$x(t) = \xi + \int_0^t f(x(s))ds. \tag{2}$$

More general initial value problems for ordinary differential equations can be reduced to this one by a change of variables. For example:

Exercise 1. Given a continuous $F\colon \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, and $(y_0, y_1, r_0) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}$, find f and ξ such that $x(t) = (y'(t + r_0), y(t + r_0), t + r_0)$ solves (1) if and only if

$$y''(r) = F(y'(r), y(r), r), \qquad y(r_0) = y_0, \quad y'(r_0) = y_1.$$

Theorem 1. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^1 near some $\xi \in \mathbb{R}^n$. There there is $T > 0$ such that there is a unique solution $x\colon [-T, T] \to \mathbb{R}^n$ to*

$$x(t) = \xi + \int_0^t f(x(s))ds. \tag{3}$$

To solve (1), we use the notation $B_r(p) = \{x \in \mathbb{R}^n \mid |x - p| < r\}$. Recall that if $f\colon B_r(p) \to \mathbb{R}^n$ is C^1 with $\|f'\| \le L$ for some constant L, then

$$|f(x) - f(y)| \le L|x - y|,$$

for all x and y in $B_r(p)$. This last inequality is called a Lipschitz condition.

Proof. Define the Picard iterates

$$x_0(t) \equiv \xi, \qquad x_{k+1}(t) = \xi + \int_0^t f(x_k(s))ds. \tag{4}$$

Take positive numbers δ, A, and L such that $|f| \le A$ and $\|f'\| \le L$ on $B_\delta(\xi)$. We require $T \le \delta/2A$ so that

$$|x_1(t) - x_0(t)| \le \int_0^t |f(\xi)|ds \le TA \le \delta/2.$$

We next require $T \le 1/2L$ so that

$$|x_2(t) - x_1(t)| \le \int_0^t |f(x_1(s)) - f(x_0(s))|ds \le TL \max_{s\in[-T,T]} |x_1(s) - x_0(s)| \le \delta/4,$$

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and, more generally,

$$|x_{k+1}(t) - x_k(t)| \le \int_0^t |f(x_k(s)) - f(x_{k-1}(s))| ds \le \max_{s\in[-T,T]} |x_k(s) - x_{k-1}(s)|/2 \le \cdots \le \delta 2^{-k-1}.$$

Thus, for $m > k$,

$$|x_m(t) - x_k(t)| \le \sum_{j=k}^{m-1} |x_{j+1}(t) - x_j(t)| \le \delta \sum_{j=k}^{m-1} 2^{-j},$$

which tends to 0 as $k \to \infty$, so x_k converges uniformly. Moreover

$$|f(x(s)) - f(x_k(s))| \le B|x(s) - x_k(s)|,$$

so that we can take the limit as $k \to \infty$ of (4) to get (3). □

The above proof gives us more, namely the following stronger (though more complicated) result.

Theorem 2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^1 near some $p \in \mathbb{R}^n$. Suppose there are positive numbers δ, A, and L such that $|f| \le A$ and $\|f'\| \le L$ on $B_{2\delta}(p)$. Then, for any $T > 0$ such that $T \le \delta/2A$ and $T \le 1/2L$, there is a unique solution $x\colon [-T,T] \times B_\delta(p) \to \mathbb{R}^n$ to*

$$x(t,\xi) = \xi + \int_0^t f(x(s,\xi))ds. \tag{5}$$

Moreover the Picard iterates

$$x_0(t,\xi) \equiv \xi, \qquad x_{k+1}(t,\xi) = \xi + \int_0^t f(x_k(s,\xi))ds, \tag{6}$$

converge uniformly to it.

The uniform convergence tells us that x is continuous in ξ; this is called *continuous dependence on the initial condition*. Thus, for every $\varepsilon > 0$ there is $\delta > 0$ such if we that know the initial condition ξ up to accuracy δ then we know $x(t)$ up to accuracy ε for all $t \in [-T,T]$. If f has additional smoothness, then so does x, allowing us to relate ε and δ:

Theorem 3. *Let $f\colon \mathbb{R}^n \to \mathbb{R}^n$ be C^2 near some $p \in \mathbb{R}^n$. There there are positive numbers δ and T such that there is a unique solution $x\colon [-T,T] \times B_\delta(p) \to \mathbb{R}^n$ to (5) and this solution is C^1.*

Proof. We have already shown that there is a unique solution if δ and T are small enough, and $\partial_t x$ is continuous by (5). To show that $\partial_{\xi_j} x$ is continuous for any j, let e_j be the unit vector in the jth coordinate direction, and let

$$y_0(t,\xi) \equiv \xi_j e_j, \qquad y_{k+1}(t,\xi) = \xi_j e_j + \int_0^t f'(x_k(s,\xi)) y_k(s,\xi)ds.$$

Thus $y_k = \partial_{\xi_j} x_k$. It is enough to show that y_k converges uniformly; then we will have

$$\lim_{k\to\infty} y_k = \lim_{k\to\infty} \partial_{\xi_j} x_k = \partial_{\xi_j} \lim_{k\to\infty} x_k = \partial_{\xi_j} x.$$

Put $z_k = (x_k, y_k)$. Then

$$z_0(t,\xi) = \zeta, \qquad z_{k+1}(t,\xi) = \zeta + \int_0^t g(z_k(s,\xi))ds, \tag{7}$$

with $\zeta = (\xi, \xi_j e_j)$ and $g(a,b) = (f(a), f'(a)b)$. We now apply Theorem 2 with f replaced by g, and with p replaced by $(p, p_j e_j)$, to conclude that z_k converges uniformly, and so in particular y_k does. □

More generally, if f is C^{K+1}, then in (7) g is C^K, so that $\partial_{\xi_j} x$ is C^{K-1}, and x is C^K. In particular, if f is C^∞, then so is x.

Moreover, if the differential equation depends smoothly on a parameter, then so does the solution. To solve

$$y'(t,\mu) = g(y,\mu), \qquad y(t_0) = y_0,$$

sove $x'(t) = (g(x_1,x_2),0)$, $x(t_0) = (y_0,\mu)$ and put $y = x_1$.

Example. Consider a falling object subject to gravity and drag. By Newton's laws,

$$m\dot{v} = mg - F(v),$$

where v is the downward velocity, m is the mass of the object, g is the acceleration of gravity, and we assume the drag force F obeys $F \ge 0$ and $F' \ge 0$. We consider the case where the mass is large, treating $\mu = 1/m$ as a small parameter, and where the object begins from rest. Then we get the inital value problem

$$\dot{v}(t,\mu) = g - \mu F(v(t,\mu)), \qquad v(0,\mu) = 0. \tag{8}$$

If F is infinitely differentiable, then this has a unique solution which is also infinitely differentiable. When $\mu = 0$ we solve by integration to get the limiting case where there is no drag:

$$v(t,0) = gt. \tag{9}$$

We ask, when μ is not zero but is small, how well does (9) approximate the true solution? We have $v(t,\mu) \to gt$ as $\mu \to 0$ because v is continuous. Also, $v(t,\mu) \le gt$ for all t and μ because $\dot{v}(t,\mu) \le g$ by (8). To get more precise information we use the mean value theorem to write

$$v(t,\mu) = v(t,0) + \mu\partial_\mu v(t,\lambda) = gt + \mu\partial_\mu v(t,\lambda),$$

where $\lambda \in [0,\mu]$. Thus gt approximates $v(\mu,t)$ up to an error which is at most linear in μ:

$$0 \ge v(t,\mu) - gt \ge \mu \min_{\lambda\in[0,\mu]} \partial_\mu v(t,\lambda).$$

For yet more precise information, we differentiate (8) with respect to μ:

$$\partial_\mu \dot{v}(t,\mu) = -F(v(t,\mu)) - \mu F'(v(t,\mu))\partial_\mu v(t,\mu), \qquad \partial_\mu v(0,\mu) = 0. \tag{10}$$

Thus $\partial_\mu \dot{v} \le -\mu F'(v)\partial_\mu v$ and so $\partial_\mu v(t,\mu) \le 0$ for all t and μ.[1] Consequently,

$$\partial_\mu \dot{v}(t,\mu) \ge -F(v(t,\mu)), \qquad \partial_\mu v(t,\mu) \ge -\int_0^t F(v(s,\mu))ds.$$

One formula for the drag force is

$$F(v) = \frac{1}{5}\rho A v^2,$$

where A is the cross-sectional area of the falling object and ρ is the density of the fluid through which it is falling; this gives a good approximation if e.g. the object is a ball the size of a bowling ball or soccer ball, falling through air, and v is between 0.15 and 30 meters per second. We obtain

$$\partial_\mu v(t,\mu) \ge -\frac{1}{5}\rho A \int_0^t v(s,\mu)^2 ds \ge -\frac{1}{5}\rho A \int_0^t g^2 s^2 ds = -\frac{1}{15}\rho A g^2 t^3,$$

and thus

$$0 \ge \frac{v(t,\mu)}{gt} - 1 \ge -\frac{\mu\rho A g t^2}{15}. \tag{11}$$

[1] Prove this, by showing that if $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable, $f(0) = 0$, and $f(x) > 0$ for some $x > 0$, then there is $y \in (0,x)$ such that $f(y) > 0$ and $f'(y) > 0$.

For a bowling ball falling through air we have have (in SI units) $\mu \le .15$, $\rho \le 1.25$, $A \le .04$, $g \le 10$, so the right hand side is $\ge -0.005t^2$. For a soccer ball we have instead $\mu \le 2.5$ and so the right hand side is $\ge -0.09t^2$. Thus in the bowling ball case we have a relative accuracy better than 0.5% for the first 1 second, and better than 2% for the first 2 seconds, and for the soccer ball it is better than 9% for the first 1 second.[2]

Exercise 2. For lower speeds and smaller balls a better formula for the drag force is

$$F(v) = 6\pi\eta R v,$$

where R is the radius of the ball and η is the dynamic viscosity of the fluid through which it is falling. What is the analogue of (11) in this case? Up to what times is the relative accuracy better than 10%, i.e. the right hand side of the analogue of (11) is $\ge -1/10$, for a falling poppy seed on Mars, where $R \le 10^{-3}$, $\eta \le 10^{-5}$, $\mu \le 10^7/3$?

We can improve accuracy by estimating errors more carefully in the above calculation, or by adding correction terms. We have

$$v(t,\mu) = v(t,0) + \mu\partial_\mu v(t,0) + \mu^2\partial_\mu^2 v(t,0)/2 + \cdots$$

and so far we have been using only the first term to approximate $v(t,\mu)$. From (10) we have

$$\partial_\mu \dot{v}(t,0) = -F(v(t,0)) = -\frac{1}{5}\rho A g^2 t^2, \qquad \partial_\mu v(0,0) = 0,$$

so that

$$\partial_\mu v(t,0) = -\frac{1}{15}\rho A g^2 t^3,$$

and

$$\begin{aligned} v(t,\mu) &= v(t,0) + \mu\partial_\mu v(t,0) + \mu^2\partial_\mu^2 v(t,0)/2 + \cdots \\ &= gt - \mu\Big(\frac{1}{15}\rho A g^2 t^3\Big) + \mu^2\partial_\mu^2 v(t,0)/2 + \cdots \end{aligned}$$

Exercise 3. A more robust formula for the drag force on a ball, applicable to a wider range of sizes and speeds, is

$$F(v) = \alpha v + \beta v^{3/2} + \gamma v^2,$$

where[3] $\alpha = 9\pi\eta R/2$, $\beta = 3\pi R^{3/2}\sqrt{\rho\eta/2}$, and $\gamma = \pi R^2\rho/5$. Find the order μ approximation, i.e. $v(t,0) + \mu\partial_\mu v(t,0)$ for this F, in terms of α, β, and γ.

The presentation of the theorems follows Chapters 2 and 4 of [Ar]. See the end of Chapter 4 for a proof that if f is C^K, then x is C^K. See Chapter III of [In], especially Sections 3.2 and 3.4, for more background on existence and uniqueness. The material on drag is adapted from [TiWe],

References

[Ar] Vladimir I. Arnold, *Ordinary Differential Equations*, 1984.
[In] E. L. Ince, *Ordinary Differential Equations*, 1926.
[TiWe] Peter Timmerman and Jacobus P. van der Weele, *On the rise and fall of a ball with linear or quadratic drag*, Amer. J. Phys, 1999.

[2]Note however that these esimtates ignore inaccuracies in our drag force formula (especially important when v is small), and other forces like buoyancy.

[3]This formula is adapted from equation (3) of [TiWe] using $\frac{1}{1+\sqrt{\mathrm{Re}}} \approx \frac{1}{\sqrt{\mathrm{Re}}} - \frac{1}{\mathrm{Re}}$.