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Following Chapter 3 of [Sp], we call a *rectangle* in $\mathbb{R}^n$ a set of the form $R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$. When R is a rectangle and $f\colon R \to \mathbb{R}$ is continuous, define $\int_R f$ to be the the supremum of the lower Riemann sums of f, which is equal to the infimum of the upper Riemann sums of f. When $A \subset \mathbb{R}^n$ is open, define

$$\int_A f = \sum_{j=1}^{\infty} \int_{R_j} \varphi_j f,$$

where φ_1, φ_2, ... are a continuous partition of unity on A with each φ_j supported in A, and each R_j is any rectangle containing the support of φ_j. The crucial properties of the sequence φ_1, φ_2, ... are that they are continuous functions from $\mathbb{R}^n \to \mathbb{R}$ and:

(1) For all $x \in \mathbb{R}^n$ and $j \in \mathbb{N}$ we have $0 \le \varphi_j(x) \le 1$.
(2) For every compact set $K \subset \mathbb{R}^n$, we have $\operatorname{supp} \varphi_j \cap K \ne \varnothing$ for only finitely many j.
(3) For all $x \in A$ we have $\sum_{j=1}^{\infty} \varphi_j(x) = 1$
(4) $\operatorname{supp} \varphi_j$ is a compact subset of A for all j.

Here $\operatorname{supp} \varphi_j$, the support of φ_j, is the closure of the set of points where $\varphi_j \ne 0$.

To construct such a partition of unity, for each integer i, let A_i be the set of $x \in A$ such that $|x| \le i$ and the distance from x to $\mathbb{R}^n \setminus A$ is $\ge 1/i$. Let $B_1 = A_1$ and for $i \ge 2$ let $B_i = A_i \setminus$interior of A_{i-1}. Then B_1, B_2, B_3 ... are a sequence of compact sets contained in A which cover A, and such that each B_i overlaps only B_{i-1} and B_{i+1}. For each $x \in B_i$, take $r(x)$ such that the open ball centered at x of radius $r(x)$ is contained in $B_{i-1} \cup B_i \cup B_{i+1}$. Then use the Heine–Borel theorem to take a sequence x_1, x_2, ... such that the open balls centered at each x_j and of radius $r(x_j)$ cover A.[1] Put $\tilde\varphi_j(x) = 0$ when $|x - x_j| \ge r(x_j)$, and $\tilde\varphi_j(x) = r(x_j) - |x - x_j|$ when $|x - x_j| \le r(x_j)$, and put

$$\varphi_j(x) = \tilde\varphi_j(x) \Big/ \sum_{m=1}^{\infty} \tilde\varphi_m(x).$$

We check that this definition is independent of the choice of partition of unity as long as $\sum \int_{R_j} \varphi_j |f|$ converges by writing

$$\sum_{j=1}^{\infty} \int_{R_j} \varphi_j f = \sum_{j=1}^{\infty} \int_{R_j} \sum_{k=1}^{\infty} \psi_k \varphi_j f = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} \int_{R_j} \psi_k \varphi_j f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{R_j} \psi_k \varphi_j f = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{S_k} \psi_k \varphi_j f,$$

where ψ_k is any other partition of unity, and each S_k any rectangle containing the support of ψ_k. The first equality follows from (3). For the second we observe that the inner sum is finite by property (2), with K replaced by $\operatorname{supp} \psi_k$. For the third use $|\int_{R_j} \varphi_j \psi_k f| \le \int_{R_j} \varphi_j |f|$ and the fact that for an absolutely convergent double sum the order of summation can be switched. For the fourth we use the fact that R_j contains the support of φ_j and S_k contains the support of ψ_k. Finally, bringing in the sum in j and using property (3) again shows that this equals $\sum \int S_k \psi_k f$.

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[1] More specifically, apply Heine–Borel to each B_i to get a finite set $x_{i,1}, \dots x_{i,L_i}$ such that the open balls centered at each $x_{i,\ell}$ of radius $r(x_{i,\ell})$ cover B_i, and then put together those finite sets into a sequence x_1, x_2 ...

Next, we check that if A is bounded and f is continuous, then

$$\sum_{j=1}^{J}\int_{R_j}\varphi_j|f| = \sum_{j=1}^{J}\int_{R}\varphi_j|f| \le \max|f|\sum_{j=1}^{J}\int_{R}\varphi_j = \max|f|\int_R\sum_{j=1}^{J}\varphi_j \le \max|f|\int_R 1,$$

and hence $\sum\int_{R_j}\varphi_j|f|$ converges since the right side is independent of J.

Finally we check that the new definition agrees with the old one for rectangles as follows. If R is a rectangle, and K is any rectangle contained in the interior of R, then

$$\Big|\int_R f - \sum_{j=1}^{J}\int_R f\varphi_j\Big| \le \max|f|\int_R 1-\sum_{j=1}^{J}\varphi_j = \max|f|\int_R\sum_{j=J+1}^{\infty}\varphi_j \le \max|f| \text{ vol }(R\setminus K) \qquad (1)$$

for J large enough, by property (2). Since K can be chosen to make the volume of $R\setminus K$ arbitarily small, this shows that $f=\sum_{j=1}^{\infty}\int_R f\varphi_j$.

As an example, let $A=\{(x,y)\in\mathbb{R}^2\colon a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$ for some numbers $a<b$ and some continuous functions $g_1<g_2$. Then, by Fubini's theorem,

$$\int_A f = \sum_{j=1}^{\infty}\int_R\varphi_j f = \sum_{j=1}^{\infty}\int_a^b\int_c^d\varphi_j(x,y)f(x,y)dydx = \sum_{j=1}^{\infty}\int_a^b\int_{g_1(x)}^{g_2(x)}\varphi_j(x,y)f(x,y)dydx$$
$$= \int_a^b\int_{g_1(x)}^{g_2(x)}\sum_{j=1}^{\infty}\varphi_j(x,y)f(x,y)dydx = \int_a^b\int_{g_1(x)}^{g_2(x)}f(x,y)dydx,$$

where $R=[a,b]\times[c,d]$ is some rectangle containing A, for the third equality we used the fact that supp $\varphi_j\subset A$, and the only tricky step is the fourth equality. For that we use the fact that, as in equation (1) above, for every $\varepsilon>0$ we can find J such that supp $\varphi_j\cap A_\varepsilon=\varnothing$ when $j\ge J$, where $A_\varepsilon=\{(x,y)\in\mathbb{R}^2\colon a+\varepsilon\le x\le b-\varepsilon,\ g_1(x)+\varepsilon\le y\le g_2(x)-\varepsilon\}$ Then

$$\Big|\int_A f - \int_a^b\int_{g_1(x)}^{g_2(x)}\sum_{j=1}^{J}\varphi_j(x,y)f(x,y)dydx\Big| \le \int_a^b\int_{g_1(x)}^{g_2(x)}h_J$$
$$\le \int_a^{a+\varepsilon}\int_{g_1(x)}^{g_2(x)}h_J + \int_{b-\varepsilon}^{b}\int_{g_1(x)}^{g_2(x)}h_J + \int_a^b\int_{g_1(x)}^{g_1(x)+\varepsilon}h_J + \int_a^b\int_{g_2(x)-\varepsilon}^{g_1(x)}h_J,$$

where $h_J=h_J(x,y)=\max|f|\sum_{j=J+1}^{\infty}\varphi_j(x,y)$. Using the fact that $h_J\le\max|f|$, we see that the right hand side goes to zero as $\varepsilon\to 0$, as desired.

References

[Sp] Michael Spivak, *Calculus on Manifolds*, 1965.