hence Theorem 6.12(*b*) implies

$$|\mathbf{y}|^2 \le |\mathbf{y}| \int |\mathbf{f}|\, d\alpha. \tag{43}$$

If $\mathbf{y} = \mathbf{0}$, (40) is trivial. If $\mathbf{y} \ne \mathbf{0}$, division of (43) by $|\mathbf{y}|$ gives (40).

RECTIFIABLE CURVES

We conclude this chapter with a topic of geometric interest which provides an application of some of the preceding theory. The case $k = 2$ (i.e., the case of plane curves) is of considerable importance in the study of analytic functions of a complex variable.

6.26 Definition A continuous mapping γ of an interval $[a, b]$ into R^k is called a *curve* in R^k. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

If γ is one-to-one, γ is called an *arc*.
If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

It should be noted that we define a curve to be a *mapping*, not a point set. Of course, with each curve γ in R^k there is associated a subset of R^k, namely the range of γ, but different curves may have the same range.

We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

$$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|.$$

The ith term in this sum is the distance (in R^k) between the points $\gamma(x_{i-1})$ and $\gamma(x_i)$. Hence $\Lambda(P, \gamma)$ is the length of a polygonal path with vertices at $\gamma(x_0)$, $\gamma(x_1), \ldots, \gamma(x_n)$, in this order. As our partition becomes finer and finer, this polygon approaches the range of γ more and more closely. This makes it seem reasonable to define the *length* of γ as

$$\Lambda(\gamma) = \sup \Lambda(P, \gamma),$$

where the supremum is taken over all partitions of $[a, b]$.

If $\Lambda(\gamma) < \infty$, we say that γ is *rectifiable*.

In certain cases, $\Lambda(\gamma)$ is given by a Riemann integral. We shall prove this for *continuously differentiable* curves, i.e., for curves γ whose derivative γ' is continuous.

6.27 Theorem *If γ' is continuous on $[a, b]$, then γ is rectifiable, and*

$$\Lambda(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Proof If $a \le x_{i-1} < x_i \le b$, then

$$|\gamma(x_i) - \gamma(x_{i-1})| = \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| \le \int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt.$$

Hence

$$\Lambda(P, \gamma) \le \int_a^b |\gamma'(t)|\, dt$$

for every partition P of $[a, b]$. Consequently,

$$\Lambda(\gamma) \le \int_a^b |\gamma'(t)|\, dt.$$

To prove the opposite inequality, let $\varepsilon > 0$ be given. Since γ' is uniformly continuous on $[a, b]$, there exists $\delta > 0$ such that

$$|\gamma'(s) - \gamma'(t)| < \varepsilon \qquad \text{if } |s - t| < \delta.$$

Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$, with $\Delta x_i < \delta$ for all i. If $x_{i-1} \le t \le x_i$, it follows that

$$|\gamma'(t)| \le |\gamma'(x_i)| + \varepsilon.$$

Hence

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} |\gamma'(t)|\, dt &\le |\gamma'(x_i)|\, \Delta x_i + \varepsilon\, \Delta x_i \\
&= \left|\int_{x_{i-1}}^{x_i} [\gamma'(t) + \gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le \left|\int_{x_{i-1}}^{x_i} \gamma'(t)\, dt\right| + \left|\int_{x_{i-1}}^{x_i} [\gamma'(x_i) - \gamma'(t)]\, dt\right| + \varepsilon\, \Delta x_i \\
&\le |\gamma(x_i) - \gamma(x_{i-1})| + 2\varepsilon\, \Delta x_i.
\end{aligned}$$

If we add these inequalities, we obtain

$$\begin{aligned}
\int_a^b |\gamma'(t)|\, dt &\le \Lambda(P, \gamma) + 2\varepsilon(b - a) \\
&\le \Lambda(\gamma) + 2\varepsilon(b - a).
\end{aligned}$$

Since ε was arbitrary,

$$\int_a^b |\gamma'(t)|\, dt \le \Lambda(\gamma).$$

This completes the proof.