set of a $\mathcal{C}^1$ function is a prototypical example. Indeed, as we now wish to make clear, there are three equivalent formulations, roughly these:

Explicit: Near each point, M is a graph over some k-dimensional coordinate plane.

Implicit: Near each point, M is the level set of some function whose derivative has maximum rank.

Parametric: Near each point, M is parametrized by some one-to-one function whose derivative has maximum rank (e.g., a parametrized curve with nonzero velocity).

We've seen that the implicit formulation arises in working with Lagrange multipliers, and the parametric formulation will be crucial for our work with integration in Chapter 8. In this brief section, we are going to make the three definitions quite precisely and then prove their equivalence in Theorem 3.1. To make our life easier in Chapter 8, we will replace the $\mathcal{C}^1$ condition with "smooth."

Definition We say $M \subset \mathbb{R}^n$ is a *k-dimensional manifold* if any one of the following three criteria holds:

1. For any $\mathbf{p} \in M$, there is a neighborhood $W \subset \mathbb{R}^n$ of $\mathbf{p}$ so that $M \cap W$ is the graph of a smooth function $\mathbf{f}\colon V \to \mathbb{R}^{n-k}$, where $V \subset \mathbb{R}^k$ is an open set. Here we are allowed to choose any k integers $1 \le i_1 < \cdots < i_k \le n$; then $\mathbb{R}^k$ is the $x_{i_1} \cdots x_{i_k}$-plane, and $\mathbb{R}^{n-k}$ is the plane of the complementary coordinates.
2. For any $\mathbf{p} \in M$, there are a neighborhood $W \subset \mathbb{R}^n$ of $\mathbf{p}$ and a smooth function $\mathbf{F}\colon W \to \mathbb{R}^{n-k}$ so that $\mathbf{F}^{-1}(\mathbf{0}) = M \cap W$ and $\text{rank}(D\mathbf{F}(\mathbf{x})) = n - k$ for every $\mathbf{x} \in M \cap W$.
3. For any $\mathbf{p} \in M$, there is a neighborhood $W \subset \mathbb{R}^n$ of $\mathbf{p}$ so that $M \cap W$ is the image of a smooth function $\mathbf{g}\colon U \to \mathbb{R}^n$ for some open set $U \subset \mathbb{R}^k$, with the properties that $\mathbf{g}$ is one-to-one, $\text{rank}(D\mathbf{g}(\mathbf{u})) = k$ for all $\mathbf{u} \in U$, and $\mathbf{g}^{-1}\colon M \cap W \to U$ is continuous. (See Figure 3.1.)

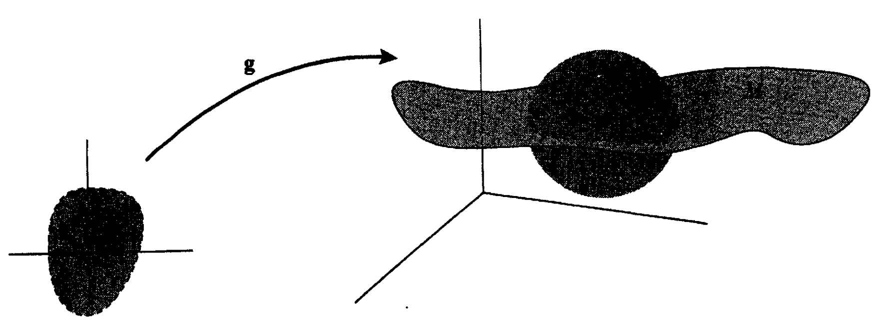

Figure 3.1

If the curious reader wonders why the last (and obviously technical) condition is included in the third definition, see Exercises 2 and 3.

Theorem 3.1 *The three criteria given in this definition are all equivalent.*

Proof The Implicit Function Theorem, Theorem 2.2, tells us precisely that (2)$\Longrightarrow$(1). And (1)$\Longrightarrow$(3) is obvious since we can set $\mathbf{g}(\mathbf{u}) = \begin{bmatrix} \mathbf{u} \\ \mathbf{f}(\mathbf{u}) \end{bmatrix}$ (where, for ease of notation, we assume here that $\mathbb{R}^k$ is the $x_1 \cdots x_k$-plane). So it remains only to check that (3)$\Longrightarrow$(2).

Suppose, as in the third definition, that we are given a neighborhood $\widetilde{W} \subset \mathbb{R}^n$ of $\mathbf{p} \in M$ so that $M \cap \widetilde{W}$ is the image of a smooth function $\mathbf{g}\colon U \to \mathbb{R}^n$ for some open set $U \subset \mathbb{R}^k$, with the properties that $\mathbf{g}$ is one-to-one, $\operatorname{rank}(D\mathbf{g}(\mathbf{u})) = k$ for all $\mathbf{u} \in U$, and $\mathbf{g}^{-1}\colon M \cap \widetilde{W} \to U$ is continuous. The last condition tells us that that if $\mathbf{g}(\mathbf{u}_0) = \mathbf{p}$, then points sufficiently close to $\mathbf{p}$ in M must map by $\mathbf{g}^{-1}$ close to $\mathbf{u}_0$; that is, all points of $M \cap \widetilde{W}$ are the image under $\mathbf{g}$ of a neighborhood of $\mathbf{u}_0$.

We may assume that $\mathbf{g}(\mathbf{0}) = \mathbf{p}$ and (renumbering coordinates in $\mathbb{R}^n$ as necessary) $D\mathbf{g}(\mathbf{0}) = \begin{bmatrix} A \\ \hline B \end{bmatrix}$, where A is an invertible $k \times k$ matrix. We define $\mathbf{G}\colon U \times \mathbb{R}^{n-k} \to \mathbb{R}^n$ by $\mathbf{G}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = \mathbf{g}(\mathbf{u}) + \begin{bmatrix} \mathbf{0} \\ \mathbf{v} \end{bmatrix}$. Since

$$D\mathbf{G}\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} = \left[\begin{array}{c|c} A & \text{O} \\ \hline B & I_{n-k} \end{array}\right]$$

is invertible (see Exercise 4.2.7), it follows from the Inverse Function Theorem, Theorem 2.1, that there are neighborhoods $V = V_1 \times V_2 \subset \mathbb{R}^k \times \mathbb{R}^{n-k}$ of $\begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}$ and $W \subset \mathbb{R}^n$ of $\mathbf{p}$ and a local (smooth) inverse $\mathbf{H}\colon W \to V$ of $\mathbf{G}$. (Shrinking W if necessary, we assume $W \subset \widetilde{W}$.) Writing $\mathbf{H}(\mathbf{x}) = \begin{bmatrix} \mathbf{H}_1(\mathbf{x}) \\ \mathbf{H}_2(\mathbf{x}) \end{bmatrix} \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we define $\mathbf{F}\colon W \to \mathbb{R}^{n-k}$ by $\mathbf{F} = \mathbf{H}_2$. Now suppose $\mathbf{F}(\mathbf{x}) = \mathbf{0}$. Since $\mathbf{x} \in W$, $\mathbf{x} = \mathbf{G}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}$ for a unique vector $\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} \in V$. Then

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}\left(\mathbf{G}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}\right) = \mathbf{H}_2\left(\mathbf{G}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}\right) = \mathbf{v},$$

so $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{v} = \mathbf{0}$, which means that $\mathbf{x} = \mathbf{g}(\mathbf{u})$. This proves that the equation $\mathbf{F} = \mathbf{0}$ defines that portion of M given by $\mathbf{g}(\mathbf{u})$ for all $\mathbf{u} \in V_1$. But because $W \subset \widetilde{W}$, we know that such points comprise all of $M \cap W$. ■

▶ EXAMPLE 1

Perhaps an explicit example will make this proof a bit more understandable. Suppose $\mathbf{g}\colon \mathbb{R} \to \mathbb{R}^3$ is given by $\mathbf{g}(u) = \begin{bmatrix} u \\ u^2 \\ u^3 \end{bmatrix}$ and M is the image of $\mathbf{g}$. We wish to write M (perhaps locally) as the level set of a function near $\mathbf{p} = \mathbf{0}$. As in the proof, we define

$$\mathbf{G}\begin{pmatrix} u \\ v_1 \\ v_2 \end{pmatrix} = \begin{bmatrix} u \\ u^2 \\ u^3 \end{bmatrix} + \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u \\ u^2 + v_1 \\ u^3 + v_2 \end{bmatrix}.$$

We can explicitly construct the inverse function

$$\mathbf{G}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{H}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} x \\ y - x^2 \\ z - x^3 \end{bmatrix}.$$

The proof tells us to define $\mathbf{F} = \mathbf{H}_2$, and, indeed, this works. M is the zero-set of the function

$$\mathbf{F}\colon \mathbb{R}^3 \to \mathbb{R}^2 \qquad \text{given by} \qquad \mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y - x^2 \\ z - x^3 \end{bmatrix}.$$

We ask the reader to carry this procedure out in Exercise 6 in a situation where it will only work locally. ◀

There are corresponding notions of the *tangent space* of the manifold M at $\mathbf{p}$. (Recall that we shall attempt to refer to the tangent *space* as a subspace, whereas the tangent *plane* is obtained by translating it to pass through the point $\mathbf{p}$.)

Definition If the manifold M is presented in the three respective forms above, then its *tangent space* at $\mathbf{p}$, denoted $T_{\mathbf{p}}M$, is defined as follows.

1. Assuming M is locally the graph of $\mathbf{f}$ with $\mathbf{p} = \begin{bmatrix} \mathbf{a} \\ \mathbf{f}(\mathbf{a}) \end{bmatrix}$, then $T_{\mathbf{p}}M$ is the graph of $D\mathbf{f}(\mathbf{a})$.
2. Assuming M is locally a level set of $\mathbf{F}$, then $T_{\mathbf{p}}M = \mathbf{N}([D\mathbf{F}(\mathbf{p})])$.
3. Assuming M is locally parametrized by $\mathbf{g}$ with $\mathbf{p} = \mathbf{g}(\mathbf{a})$, then $T_{\mathbf{p}}M$ is the image of the linear map $D\mathbf{g}(\mathbf{a})\colon \mathbb{R}^k \to \mathbb{R}^n$.

Once again, we need to check that these three recipes all give the same k-dimensional subspace of $\mathbb{R}^n$. The ideas involved in this check have all emerged already in the preceding chapters. Since (1) is a special case of (3) (why?), we need only check that $\mathbf{N}([D\mathbf{F}(\mathbf{p})]) = \text{image}(D\mathbf{g}(\mathbf{a}))$. Note that both of these are k-dimensional subspaces because of our rank conditions on $\mathbf{F}$ and $\mathbf{g}$. So it suffices to show that $\text{image}(D\mathbf{g}(\mathbf{a})) \subset \mathbf{N}([D\mathbf{F}(\mathbf{p})])$. But this is easy: The function $\mathbf{F}\circ\mathbf{g}\colon U \to \mathbb{R}^{n-k}$ is identically $\mathbf{0}$, so, by the chain rule, $D\mathbf{F}(\mathbf{p})\circ D\mathbf{g}(\mathbf{a}) = O$, which says precisely that any vector in the image of $D\mathbf{g}(\mathbf{a})$ is in the kernel of $D\mathbf{F}(\mathbf{p})$.

▶ EXERCISES 6.3

*1. Show that the set $X = \left\{\begin{bmatrix} x \\ y \end{bmatrix} : y = |x|\right\}$ is not a 1-dimensional manifold, even though the function $\mathbf{g}(t) = \begin{bmatrix} t^3 \\ |t^3| \end{bmatrix}$ gives a $\mathcal{C}^1$ "parametrization" of it. What's going on?

2. Show that the parametric curve $\mathbf{g}(t) = \begin{bmatrix} \cos 2t \cos t \\ \cos 2t \sin t \end{bmatrix}$, $t \in (-\pi/2, \pi/4)$, is not a 1-dimensional manifold. (Hint: Stare at Figure 3.2.)

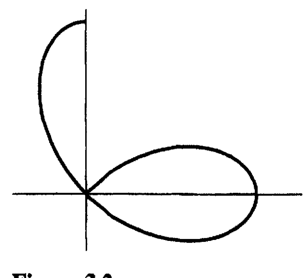

Figure 3.2

3. Consider the following union of parallel lines:

$$X = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = q \text{ for some } q \in \mathbb{Q} \right\} \subset \mathbb{R}^2.$$

Is X a 1-dimensional manifold? (Here $\mathbb{Q}$ denotes the set of rational numbers.)

4. Is the union of the hyperbola $xy = 1$ and its asymptote $y = 0$ a 1-dimensional manifold? Give your reasoning.

5. Show the equivalence of the three definitions for each of the following 1-dimensional manifolds:

*(a) parametric curve $\begin{bmatrix} t^2 \\ t \\ t^4 \end{bmatrix}$

(b) parametric curve $\begin{bmatrix} \cos t \\ 3 \sin t \end{bmatrix}$

(c) implicit curve $x^2 + y^2 = 1$, $x^2 + y^2 + z^2 = 2x$

(d) implicit curve $x^2 + y^2 = 1$, $z^2 + w^2 = 1$, $xz + yw = 0$

6. Suppose $\mathbf{g}\colon \mathbb{R} \to \mathbb{R}^3$ is given by $\mathbf{g}(u) = \begin{bmatrix} u + u^2 \\ u^2 \\ u^3 \end{bmatrix}$. Let M be the image of $\mathbf{g}$.

(a) Show that $\mathbf{g}$ is globally one-to-one.

(b) Following the proof given of Theorem 3.1, find a neighborhood W of $\mathbf{0} \in \mathbb{R}^3$ and $\mathbf{F}\colon W \to \mathbb{R}^2$ so that $M \cap W = \mathbf{F}^{-1}(\mathbf{0})$.

7. Show the equivalence of the three definitions for each of the following 2-dimensional manifolds:

(a) implicit surface $x^2 + y^2 = 1$ (in $\mathbb{R}^3$)

(b) implicit surface $x^2 + y^2 = z^2$ (in $\mathbb{R}^3 - \{\mathbf{0}\}$)

*(c) parametric surface $\begin{bmatrix} u \cos v \\ u \sin v \\ v \end{bmatrix}$, $u > 0$, $v \in \mathbb{R}$

(d) parametric surface $\begin{bmatrix} \sin u \cos v \\ \sin u \sin v \\ \cos u \end{bmatrix}$, $0 < u < \pi$, $0 < v < 2\pi$