**MA 504 final exam review problems**

Version as of December 6th.

The final will be on Tuesday, December 13th, from 8 to 10 am in UNIV 101. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $|f(x) - f(y)| \geq |x - y|$ for all $x, y \in \mathbb{R}$. Prove that $f$ is surjective.

2. For each of the following functions, determine the pointwise limit $f(x)$ on the indicated interval, and decide whether the convergence is uniform. If the convergence is uniform, find a sequence of real numbers $B_n \to 0$ such that $|f_n(x) - f(x)| \leq B_n$ for all $x$ in the interval. If it isn't, find $\varepsilon > 0$ and a sequence $x_n$ such that $|f_n(x_n) - f(x_n)| \geq \varepsilon$ for all $n$.
   (a) $f_n(x) = n^{-1} \sin x$ on $\mathbb{R}$.
   (b) $f_n(x) = e^{nx}$ on $(-\infty, -1)$.
   (c) $f_n(x) = e^{-nx^2}$ on $[-10, 10]$.
   (d) $f_n(x) = nx/(1 + n + x)$ on $[0, 10]$.
   (e) $f_n(x) = nx/(1 + n + x)$ on $[0, \infty)$.

3. Let $f\colon [a, b] \to [c, d]$ be continuously differentiable and such that $f$ and $f'$ are both monotonic. Give an upper bound, as small as you can, on the arclength of the graph of $f$ in terms of $a$, $b$, $c$, and $d$.

   *Hint:* Prove that $\sqrt{\alpha + \beta} \leq \sqrt{\alpha} + \sqrt{\beta}$ for suitable $\alpha$ and $\beta$ and use the integral formula for arclength.

4. Is the sequence of functions $f_n\colon \mathbb{R} \to \mathbb{R}$ defined by
$$f_n(x) = \sin(n + x) + \frac{1}{1 + \frac{1}{\sqrt{n+1}} \cos^2(n!x)}$$
   equicontinuous?

5. Let $K$ be a compact metric space, and let $\{f_1,\ f_2,\ \ldots\}$ be a uniformly bounded equicontinuous family of functions $K \to \mathbb{C}$. For every positive integer $n$ and $x \in K$, let
$$g_n(x) = \max\{|f_1(x)|, \ldots, |f_n(x)|\}.$$
   Prove that the sequence $g_1,\ g_2,\ \ldots$ converges uniformly on $K$.

6. Find a real number $\alpha > 0$, as large as possible, such that for any $a, b \in \mathbb{R}$ with $a < b$ and any twice differentiable $f\colon \mathbb{R} \to \mathbb{R}$ satisfying $f'(a) = f'(b) = 0$, there is a point $c \in [a, b]$ such that
$$|f''(c)| \geq \alpha \frac{f(b) - f(a)}{(b - a)^2}.$$

(Incidentally, this means that if a particle covers distance $d$ in time $t$, starting and ending at rest, then at some point its acceleration must be at least $\alpha\, d/t^2$.)

*Hint:* Use Theorem 5.15 on the intervals $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$ and subtract the resulting equations.

7. Exercises 11.7.4–11.7.8 from `https://www.jirka.org/ra/html/sec_stoneweier.html`. (These are numerous but repetitive).

8. Let $k$ be a positive integer, and let $K \subset \mathbb{R}^k$ be compact, and let $\alpha > 0$ be given. Let $C(K)$ be the metric space of all continuous functions $K \to \mathbb{C}$, with the distance between two functions $f_1,\, f_2 \in C(K)$ defined to be $\max_{p \in K} |f_1(p) - f_2(p)|$. Let $A$ be the set of all functions $g \colon K \to \mathbb{C}$ for which there is a constant $C$ such that $|g(p) - g(q)| \leq C|p - q|^\alpha$ for any $p$ and $q$ in $K$. Prove that $A$ is dense in $C(K)$ if $\alpha \leq 1$, or if $K$ contains only finitely many points, but not if $\alpha > 1$ and $k = 1$ and $K$ contains an interval.

9. Let $X$ be a metric space, let $F \subset X$ be a finite set, and let $E \subset X$ be a countable set. Suppose every sequence in $E$ has a subsequence converging to a point in $F$. Prove that the closure of $E$ is compact.