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MA 504
Fall 2022

Homework 1

Due Friday, September 2nd at the beginning of class. There are some hints on the last page. Please let me know if you have a question or find a mistake.

1. Prove that a least upper bound is always unique, i.e. if S is an ordered set and $E \subset S$ is bounded above, and if both α and α' are least upper bounds in the sense of Rudin's definition 1.8, then $\alpha = \alpha'$.

2. Let $E \subset \mathbb{R}$ and let α be a real number. Prove that the following are equivalent.
   (a) For every real x, we have $x < \alpha$ if and only if there is $y \in E$ such that $x < y$.
   (b) $\alpha = \sup E$.
   (c) For every real x, we have $x \geq \alpha$ if and only if $x \geq y$ for all $y \in E$.

3. Let[1]
$$f(p) = \frac{1}{2}\left(p + \frac{2}{p}\right).$$
   (a) Prove that if $p \in \mathbb{Q}$ and $p > 0$, then $f(p)^2 > 2$.
   (b) Prove that if $p \in \mathbb{Q}$ and $p > 0$ and $p^2 > 2$, then $f(p) < p$.
   (c) You don't have to hand anything in for this, but you might like to think about what it would take to conclude from the above that the set $\{x \in \mathbb{Q} \colon x^2 \leq 2\}$ has no least upper bound in $\mathbb{Q}$.

[1]This function is the basis of the first algorithm to be discovered for computing $\sqrt{2}$. For *any* starting positive number p, applying f repeatedly leads to a sequence of numbers converging rapidly to $\sqrt{2}$. We will prove this later but if you haven't seen this before I recommend trying it.

Hint for problem 1: Show that $\alpha < \alpha'$ and $\alpha' < \alpha$ both lead to contradictions.

Hint for problem 2: Show that (a) and (c) are logically equivalent. Then use the definitions to check that (b) and (c) are equivalent.

Hint for problem 3: For part (a), factor $f(p)^2 - 2$. For part (b), examine the two numbers whose average is taken to obtain $f(p)$.