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MA 504
Fall 2022

Homework 4

Due October 7th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by continuing the following pattern:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4},$$

$$1 + \frac{1}{2} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}.$$

   (b) Use the result of part (a) to to find an unbounded increasing sequence $a_1, a_2, \ldots$ such that $\sum_{n=1}^{2^k} \frac{1}{n} \geq a_k$ for all positive integers k.

2. (a) Prove that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by proving that if $n > 1$ then

$$\frac{1}{n^2} < \frac{1}{n(n-1)},$$

   and using partial fraction decomposition to simplify the partial sums of $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$.

   (b) What upper bound on the value of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is implied by the calculation of (a)?

   Note: See also Theorems 3.27 and 3.28 of Rudin for more insight into the above two problems, but don't invoke them here.

3. Let $a_1,\ a_2,\ \ldots$ be a sequence of positive numbers which tends to zero but such that $\sum_{n=1}^{\infty} a_n$ diverges. Let $A_1,\ A_2,\ \ldots$ be the sequence of partial sums

$$A_n = \sum_{k=1}^{n} a_k,$$

   and let $b_1 = \sqrt{A_1}$, $b_{n+1} = \sqrt{A_{n+1}} - \sqrt{A_n}$ for $n \geq 1$. Show that

$$\lim_{n\to\infty} \frac{b_n}{a_n} = 0,$$

   but that $\sum_{n=1}^{\infty} b_n$ is still divergent. In this sense there is no 'smallest' divergent series, and one can similarly show that there is no 'largest' convergent one.

   Hint: Show that if α and β are positive, then $\sqrt{\alpha} - \sqrt{\beta} = \frac{\alpha-\beta}{\sqrt{\alpha}+\sqrt{\beta}}$.

4. Let $s_1,\ ,s_2,\ \ldots$ be a bounded sequence of real numbers, and define s^* as in Rudin's Definition 3.16. Let $b_n = \sup\{s_n,\ s_{n+1},\ \ldots\}$, and let $\alpha = \lim_{n\to\infty} b_n$. Prove that $s^* = \alpha$. Do not use

lim sup notation in this problem since this amounts to showing that the definition of limsup we used in class agrees with the definition in the book.

Hint: With E as in Rudin's Definition 3.16, show that α is an upper bound for E directly from the definition. Next show that if $\beta < \alpha$, then β is not an upper bound for E, by showing that there are infinitely many s_n obeying $\frac{\alpha+\beta}{2} \leq s_n$.