Kiril Datchev
MA 504
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Homework 5

Due October 14th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

There are some hints on the second page.

1. Let X be a complete metric space, let $\lambda \in [0, 1)$, and let $f: X \to X$ obey $d(f(x), f(y)) \leq \lambda d(x, y)$ for all $x,\ y \in X$. Prove that there is a unique $x \in X$ such that $f(x) = x$. (One says 'a contraction on a complete metric space has a unique fixed point').

2. Let X be a compact metric space and let $f: X \to X$ obey $d(f(x), f(y)) < d(x, y)$ for all $x,\ y \in X$ such that $x \neq y$. Prove that there is a unique $x \in X$ such that $f(x) = x$.

3. Let $X = [0, 1]$ with $d(x, y) = |x - y|$ and $f: X \to X$ be given by $f(x) = x - x^2$. Prove that the assumptions of Problem 2 are satisfied but not the assumptions of Problem 1.

4. Let $f: [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) > 0$ for all $x \geq 0$ and such that

$$\lim_{x \to \infty} f(x) = 0.$$

   (a) Prove that f is bounded on $[0, \infty)$.

   (b) Prove that f has a maximum on $[0, \infty)$.

   (c) Prove that f has no minimum on $[0, \infty)$.

   (d) Prove that f is uniformly continuous.

Hints:

1. Prove that, for any $x_0 \in X$, the sequence given by $x_{n+1} = f(x_n)$ for all $n \in \{1, 2, \dots\}$ is Cauchy.
2. Prove that the function $x \mapsto d(x, f(x))$ is continuous and has a minimum value of zero.
3. Observe that $d(f(x), f(y))/d(x, y)$ simplifies nicely.
4. (a) Consider separately the intervals $[0, N]$ and (N, ∞), where N is a strategically chosen large number.

   (b) Use the same approach as in part (a), but explain how to choose N in such a way that the maximum is attained on $[0, N]$.

   (c) For any $a \geq 0$, prove that there is x such that $f(x) < f(a)$.