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MA 504
Fall 2022

Homework 7

Due November 11th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

There are some hints on the second page.

1. Exercise 1 from page 138.
2. Exercise 2 from page 138.
3. Let $x > 0$, let $n \in \mathbb{N} \cup \{0\}$, and let $f \colon [0, x] \to \mathbb{R}$ be $n+1$ times differentiable with $f^{(n+1)}$ integrable. Prove that
$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + I_n(x),$$
where
$$I_n(x) = \frac{x^{n+1}}{n!}\int_0^1 (1-t)^n f^{(n+1)}(tx)dt = \frac{1}{n!}\int_0^x (x-x')^n f^{(n+1)}(x')dx'.$$
4. Let $f \colon [0,1] \to [0,1]$ be continuously differentiable with nonincreasing derivative. Prove that the arclength of the graph of f is at most 3.

Hints:

1. Prove that $L(P, f, \alpha) = 0$ regardless of P and that for any ε you can find a P such that $U(P, f, \alpha) < \varepsilon$.
2. Prove that if $f(x_0) > 0$ for some $x_0 \in [a, b]$, then there is $\delta > 0$ such that $f(x) \geq f(x_0)/2$ in a δ-neighborhood of x_0, and hence you can find a P such that $L(P, f) > 0$.
3. Use mathematical induction and integration by parts.[1]
4. Draw a picture of $\Lambda(P, \gamma)$ for various choices of f and P, and try to choose f and P so as to make $\Lambda(P, \gamma)$ as large as possible.

[1]Note by the way that Theorem 5.15 in Rudin uses the mean value theorem to prove another version of Taylor's theorem under slightly weaker hypotheses, but this version has the advantage of giving a more explicit remainder. Of course the case $x < 0$ follows by applying the result to $g(x) = f(-x)$, and an expansion near $a \neq 0$ follows by taking $g(x) = f(a + x)$.