Kiril Datchev
MA 504
Fall 2022

Homework 8

Due November 18th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

There are some hints on the second page.

1. Prove that a compact metric space K has a countable dense subset.

2. For each of the following functions, determine the pointwise limit $f(x)$ on the indicated interval, and decide whether the convergence is uniform. If the convergence is uniform, find a sequence of real numbers $B_n \to 0$ such that $|f_n(x) - f(x)| \le B_n$ for all x in the interval. If it isn't, find $\varepsilon > 0$ and a sequence x_n such that $|f_n(x_n) - f(x_n)| \ge \varepsilon$ for all n.
   (a) $f_n(x) = x^{1/n}$ on $[0, 1]$.
   (b) $f_n(x) = n^{-1}e^{-x^2}$ on $\mathbb{R}$.
   (c) $f_n(x) = x^n - x^{2n}$ on $[0, 1]$.

3. Let
$$f(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n^2(1 + x^n)}.$$
   (a) Prove that f is continuous on $[0, \infty)$.
   (b) Evaluate
$$\int_0^1 f(x)\, dx$$
   in terms of the function
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

4. Exercise 15 from page 168.

Hints:

1. Prove that for any positive integer n, there is a finite set of points $x_{n,1}, \ldots, x_{n,M_n}$ such that for each $x \in K$ there is $j \in \{1, \ldots M_n\}$ such that $d(x, x_{n,M_j}) < 1/n$.
2. For part (c), find the maximum of f_n.
3. Estimate the fraction in a different way depending on whether $x \leq 1$ or $x \geq 1$.
4. Substitute x/n for x and y/n for y in the definition of equicontinuity.