MA 504 first midterm review problems

Version as of September 22nd.

The first midterm will be in class on Monday, September 26th. No notes or electronic devices allowed. In your proofs you may use any result from Rudin provided you state it carefully, including all the hypotheses, and provided it doesn't solve the whole problem in one line. (If it solves the whole problem in one line, then you are being asked to reprove that result.) Most of the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Determine if each of the following statements is true or false. If true, give a proof. If false, give a counterexample.
   (a) If $E \subset \mathbb{R}$ is bounded and nonempty, and $\alpha = \sup E$, then for every $\varepsilon > 0$ there is $x \in E$ such that $x \in [\alpha - \varepsilon, \alpha]$.
   (b) If $E \subset \mathbb{R}$ is bounded and nonempty, and $\alpha = \sup E$, then for every $\varepsilon > 0$ there is $x \in E$ such that $x \in (\alpha - \varepsilon, \alpha)$.
2. Let $S_1, S_2, \dots$ be a sequence of nonempty sets of negative real numbers, and let $S = \bigcup_{n=1}^{\infty} S_n$. Let $\alpha = \sup S$ and $\alpha_n = \sup S_n$ for every n. Prove that $\alpha = \sup\{\alpha_1, \alpha_2, \dots\}$.
3. Let X be a metric space, and $E \subset F \subset X$. Prove that $E^\circ \subset F^\circ$, and $\overline{E} \subset \overline{F}$.
4. Let X be a metric space, let $E \subset X$, and let $p \in X$.
   (a) Prove that p is a limit point of E if and only if every neighborhood of p contains infinitely many points from E.
   (b) Define the *distance* from p to E, denoted $d(p, E)$, by $d(p, E) = \inf\{d(p, q) \colon q \in E\}$. Prove that $p \in \overline{E}$ if and only if $d(p, E) = 0$.
   (c) We say p is a *boundary point* of E if every neighborhood $N_r(p)$ contains points of both E and E^c. The set of boundary points of E is denoted ∂E. Prove that $E^\circ = E - \partial E$ and $\overline{E} = E^\circ \cup \partial E$. Prove that E is closed if and only if it contains all of its boundary points.
5. Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open. Prove that $\{(x, y) \in \mathbb{R}^{m+n} \colon x \in U \text{ and } y \in V\} \subset \mathbb{R}^{m+n}$ is open.
6. Let X be a metric space and let $K \subset X$. Prove that K is compact if and only if $\cap_{\alpha \in A} F_\alpha \neq \varnothing$ for all families $(F_\alpha)_{\alpha \in A}$ of closed subsets of K such that $F_{\alpha_1} \cap \dots \cap F_{\alpha_n} \neq \varnothing$ for all finite subsets $\{\alpha_1, \dots, \alpha_n\} \subset A$.
7. Let $U \subset \mathbb{R}^2$ be an open set such that $\{x \in \mathbb{R}^2 \colon |x| \le 1\} \subset U$. Prove there is $\varepsilon > 0$ such that $\{x \in \mathbb{R}^2 \colon |x| \le 1 + \varepsilon\} \subset U$.
8. Let A and B be given positive real numbers. Let $a_0 = A$ and for each positive integer n let
$$a_{n+1} = \frac{1}{2}\Big(a_n + \frac{B}{a_n}\Big).$$
   Prove that the sequence $a_0,\ a_1,\ a_2,\ \dots$ converges and find its limit.