4.1 DOUBLE INTEGRALS

The main theorems about integrating functions of two variables are important and conceptually clear, but difficult to formulate and prove. In fact, the first really good theory was not developed until recently, by Henri Lebesgue (1902) and Guido Fubini (1910), to mention two of the most famous names. We will begin with the general ideas, and make only a few steps toward formulating and proving the theorems.

Suppose f is a function defined in a set S of the plane, and $f \geq 0$ on S. Then the double integral $\iint_S f$ means, intuitively, the volume of the region lying over S and under the graph of f.

Example 1. S is the unit disk $\{P: |P| \leq 1\}$, and $f(x,y) = \sqrt{1 - x^2 - y^2}$. The region lying over S and under the graph of f is the upper half of the unit ball $\{(x,y,z): x^2 + y^2 + z^2 \leq 1\}$; hence

$$\iint_S f = \tfrac{1}{2}\left(\tfrac{4}{3}\pi 1^3\right) = \tfrac{2}{3}\pi.$$

(See Fig. 1.)

Example 2. S is the square $\{(x,y): 0 \leq x \leq 1, 0 \leq y \leq 1\}$, and $f(x,y) = y$. The region lying over S and under the graph of f is a wedge, a unit cube sliced in half, so $\iint_S f = \frac{1}{2}$. (See Fig. 2.)

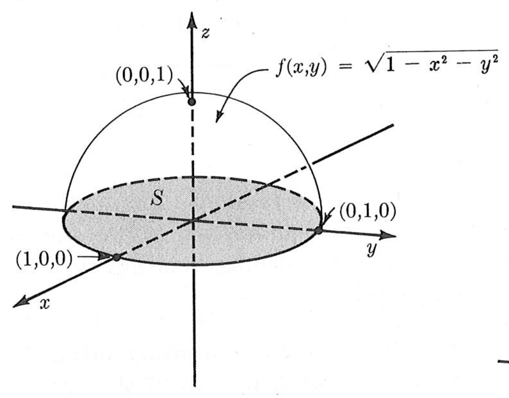


FIGURE 4.1

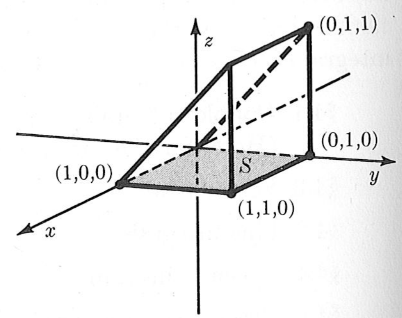


FIGURE 4.2

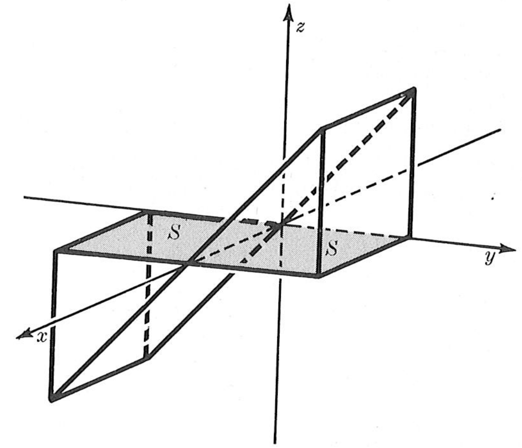


FIGURE 4.3

When f takes negative values in part of S, then $\iint_S f$ means, intuitively, the volume lying under the graph of f and over S, *minus* the volume lying over the graph and under S. In other words, volumes above the xy plane are taken as positive, and those below the xy plane as negative.

For example, if S is the rectangle $\{(x,y): -1 \leq x \leq 1, 0 \leq y \leq 1\}$, and $f(x,y) = y$, then $\iint_S f$ is the difference in the volumes of two congruent wedges, so $\iint_S f = 0$. (See Fig. 3.)

Obviously, simple pictures like these can evaluate only very special integrals. A much more general method is suggested by the formula for a volume of revolution. Recall that if we take the graph of a function g on an interval $[a,b]$ and revolve it about the x axis, we enclose a volume

$$V = \int_a^b \pi g(x)^2\, dx.$$

Here, $\pi g(x)^2$ is the area of a cross section of the enclosed volume, and the volume itself is the integral of these cross-sectional areas (Fig. 4).

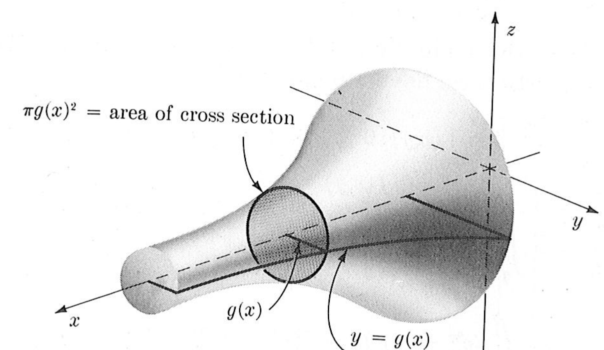


FIGURE 4.4

Now consider a function $f \geq 0$ defined on a rectangle

$$S = \{(x,y): a \leq x \leq b,\ c \leq y \leq d\};$$

then $\iint_S f$ is the volume shown in Fig. 5. Let $A(x)$ denote the area of a typical cross section by a plane perpendicular to the x axis at $(x,0,0)$. By analogy with volumes of revolution, the volume $\iint_S f$ should be

$$\iint_S f = \int_a^b A(x)\, dx. \tag{1}$$

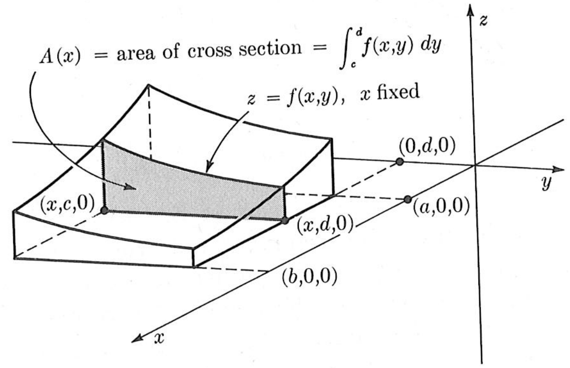


FIGURE 4.5

Further, for any fixed x, the cross-sectional area $A(x)$ is the area under the graph of a function of one variable, $g(y) = f(x,y)$, so

$$A(x) = \int_c^d f(x,y)\, dy. \tag{2}$$

Putting (2) into (1), we get

$$\iint_S f = \int_a^b \left[\int_c^d f(x,y)\, dy\right] dx. \tag{3}$$

This formula reduces the evaluation of $\iint_S f$ to the evaluation of two ordinary integrals, a question we have already studied.

Example 3. $S = \{(x,y): 0 \leq x \leq 1, 1 \leq y \leq 2\}$

$$f(x,y) = xe^{xy}$$

$$\iint_S f = \int_0^1 \left[\int_1^2 xe^{xy}\, dy\right] dx = \int_0^1 [e^{xy}]_{y=1}^{y=2}\, dx$$

$$= \int_0^1 (e^{2x} - e^x)\, dx = [\tfrac{1}{2}e^{2x} - e^x]_0^1 = \tfrac{1}{2}e^2 - e + \tfrac{1}{2}.$$

The same method can be applied when S is bounded by the graphs of two functions $\varphi_1(x)$ and $\varphi_2(x)$ on an interval $[a,b]$,

$$S = \{(x,y): a \leq x \leq b,\ \varphi_1(x) \leq y \leq \varphi_2(x)\}, \tag{4}$$

where we assume that $\varphi_1 \leq \varphi_2$ on the interval $[a,b]$. As shown in Fig. 6, a typical cross section perpendicular to the x axis has area

$$A(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy,$$

so the total volume is

$$\iint_S f = \int_a^b A(x)\, dx = \int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\, dy\right]. \tag{5}$$

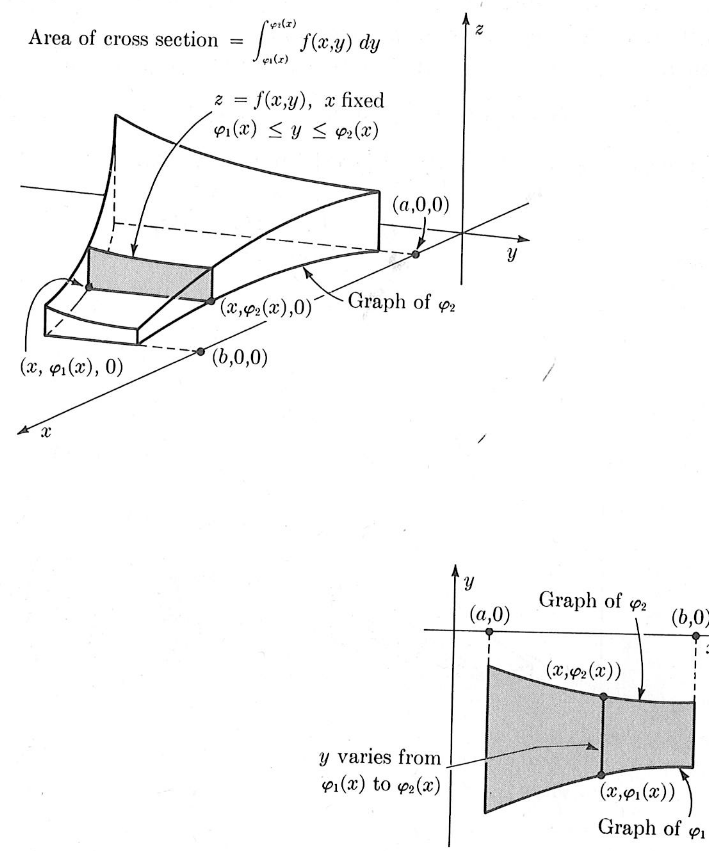


FIGURE 4.6

Example 4. For the double integral illustrated in Fig. 1 we have

$$S = \{(x,y): -1 \le x \le 1, -\sqrt{1-x^2} \le y \le \sqrt{1-x^2}\} \quad \text{(the unit disk)}$$

$$f(x,y) = \sqrt{1-x^2-y^2},$$

$$\iint_S f = \int_{-1}^{1}\left[\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dy\right] dx$$

$$= \int_{-1}^{1}\left[\frac{1-x^2}{2}\arcsin\frac{y}{\sqrt{1-x^2}} + \frac{1}{2}y\sqrt{1-x^2-y^2}\right]_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}} dx$$

$$= \int_{-1}^{1}\frac{\pi}{2}(1-x^2)\, dx = \frac{\pi}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1} = \frac{\pi}{2}\cdot\frac{4}{3} = \frac{2\pi}{3}.$$

(Here we used the integral $\int \sqrt{a^2-y^2}\, dy = \frac{a^2}{2}\arcsin\frac{y}{a} + \frac{1}{2}y\sqrt{a^2-y^2}$.)

The restriction $f \ge 0$ was made only to simplify the picture; the same method works, regardless of the sign of f. And, of course, we could just as well take sections perpendicular to the y axis (Fig. 7): if

$$S = \{(x,y): \psi_1(y) \le x \le \psi_2(y),\ c \le y \le d\}, \tag{6}$$

where $\psi_1 \le \psi_2$ on the interval $[c,d]$, then a typical cross section has area

$$A(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx,$$

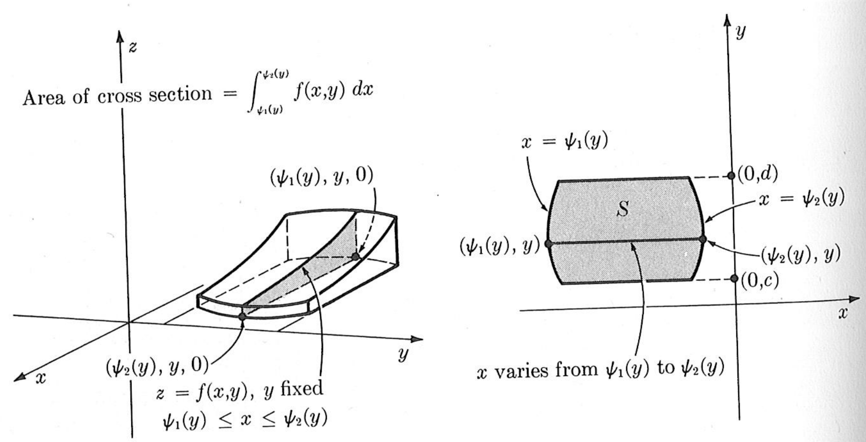


FIGURE 4.7

and

$$\iint_S = \int_c^d A(y)\, dy = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\, dx\right] dy. \tag{7}$$

In both (5) and (7) there is an "inner integral" (the one in square brackets) and an "outer integral"; the inner integral has variable limits, and the outer integral has constant limits. When the inner integral is with respect to x (as in (7)), its limits depend on y, and in the evaluation of $\int f(x,y)\, dx$ we keep y constant. When the inner integral is with respect to y (as in (5)), its limits depend on x, and in $\int f(x,y)\, dy$ we keep x constant.

Example 5. $S = \{(x,y): 0 \le y \le x^2, 0 \le x \le 2\}$ (see Fig. 8(a))

$$f(x,y) = x + y.$$

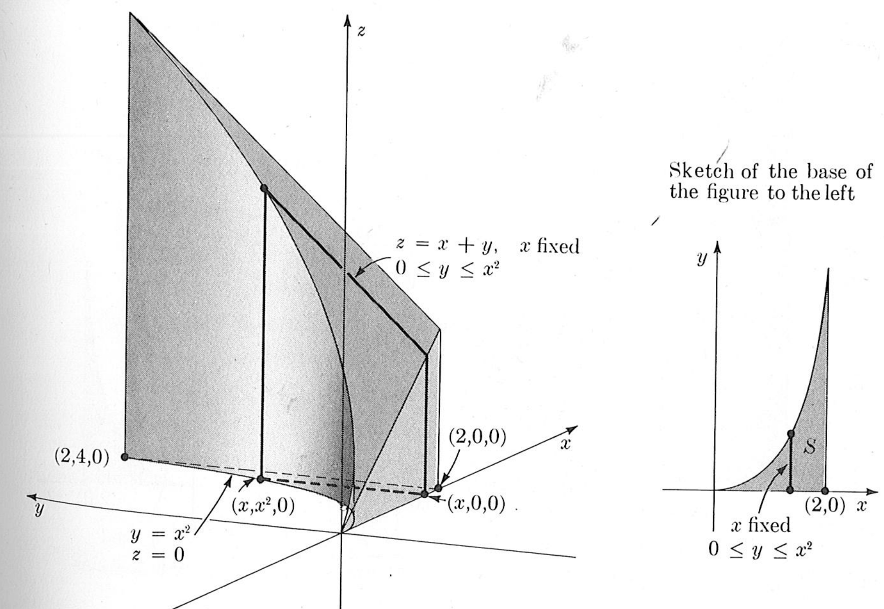


FIGURE 4.8 (a)

The set S is given in the form (4), with $\varphi_1 = 0$ and $\varphi_2 = x^2$, so we have, as in (5),

$$\iint_S f = \int_0^2 \left[\int_0^{x^2} (x+y)\,dy \right] dx = \int_0^2 \left[xy + \frac{y^2}{2} \right]_{y=0}^{y=x^2} dx$$

$$= \int_0^2 \left[x^3 + \frac{x^4}{2} \right] dx = \left[\frac{x^4}{4} + \frac{x^5}{10} \right]_0^2 = \frac{36}{5}.$$

The same set can also be described in the form (6),

$$S = \{ (x,y) \colon \sqrt{y} \le x \le 2, 0 \le y \le 4 \} \qquad \text{(see Fig. 8(b))},$$

so

$$\iint_S f = \int_0^4 \left[\int_{\sqrt{y}}^2 (x+y)\,dx \right] dy = \int_0^4 \left[\frac{x^2}{2} + xy \right]_{x=\sqrt{y}}^{x=2} dy$$

$$= \int_0^4 \left(2 + 2y - \frac{y}{2} - y^{3/2} \right) dy$$

$$= \left[2y + y^2 - \frac{y^2}{4} - \frac{2}{5} y^{5/2} \right]_0^4 = \frac{36}{5}.$$

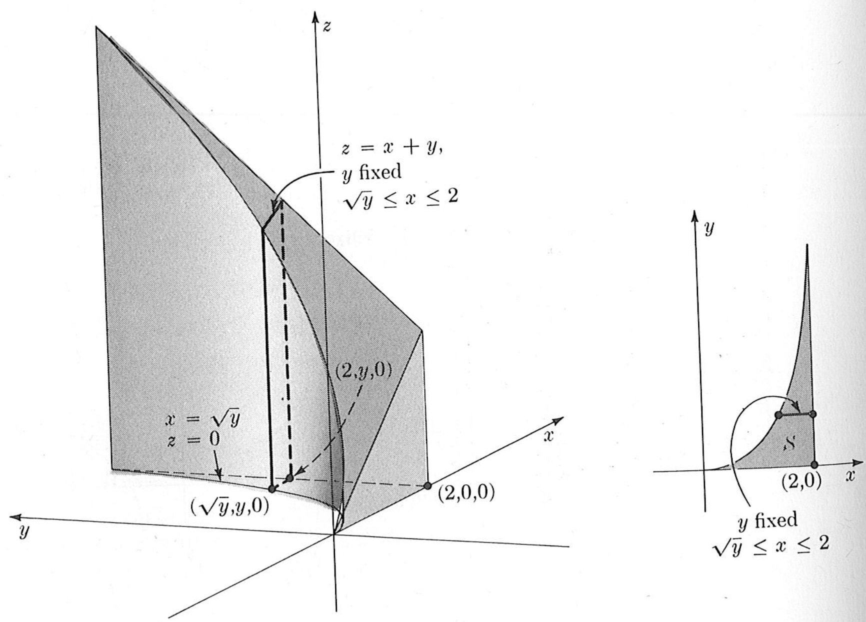


FIGURE 4.8 (b)

We have written the "repeated integral"

$$\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx$$

with square brackets to show clearly how it is to be evaluated; first take

$$\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy,$$

and then integrate the result with respect to x. Usually it is written simply

$$\int_a^b \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy\,dx,$$

with the understanding that the "inner integral" is to be evaluated first.

This whole discussion has been based on an analogy and a few pictures. Two important theorems give these pictures the necessary rigorous basis.

Theorem 1. *Let f be continuous on the set*

$$S = \{ (x,y) : a \le x \le b,\ \varphi_1(x) \le y \le \varphi_2(y) \}, \qquad (4)$$

where φ_1 and φ_2 are continuous on $[a, b]$ and $\varphi_1 \le \varphi_2$. Let

$$G(x) = \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy.$$

Then G is continuous on $[a,b]$.

Similarly, if S is the set

$$S = \{ (x,y) : \psi_1(y) \le x \le \psi_2(y),\ c \le y \le d \}, \qquad (6)$$

where ψ_1 and ψ_2 are continuous and $\psi_1 \le \psi_2$, then

$$H(y) = \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

is continuous on $[c,d]$.

Theorem 2. *If the same set S is described both by* (4) *and by* (6), *then*

$$\int_a^b \left[\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy \right] dx = \int_c^d \left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx \right] dy. \qquad (8)$$

If these theorems could be proved intelligibly in a page or two, we would do it, but in fact they are by-products of a thorough investigation of double integration, as well as the deeper properties of continuous functions of two variables. Rather than give "ad hoc" direct proofs of such special results, we refer you to the general theory as expounded in the references listed at the end of this section.

The common value of the two integrals in (8) is called the *double integral* of f over S, denoted

$$\iint_S f \qquad \text{or} \qquad \iint_S f(x,y)\, dx\, dy.$$

Since we are defining the double integral by (8), the linearity property

$$\iint_S (\alpha f + \beta g) = \alpha \iint_S f + \beta \iint_S g \qquad (\alpha \text{ and } \beta \text{ are constants})$$

follows from the linearity of the single integral (Problem 7). Similarly,

$$f \le g \;\Rightarrow\; \iint_S f \le \iint_S g.$$

Theorems 1 and 2 have an important corollary, *Leibniz' rule* for differentiating under the integral sign:

Theorem 3. *Suppose $D_1 f(t,y)$ is continuous on the rectangle*

$$R = \{(t,y) : a \le t \le b,\, c \le y \le d\}.$$

Then the function

$$G(t) = \int_c^d f(t,y)\, dy$$

has the derivative

$$G'(t) = \int_c^d D_1 f(t,y)\, dy.$$

Proof. Consider $f(t,y)$ as a function of t, with y held fixed. By the fundamental theorem of calculus

$$f(t,y) = f(a,y) + \int_a^t D_1 f(x,y)\, dx;$$

hence

$$G(t) = \int_c^d f(t,y)\, dy = \int_c^d f(a,y)\, dy + \int_c^d \left[\int_a^t D_1 f(x,y)\, dx\right] dy$$

$$= \int_c^d f(a,y)\, dy + \int_a^t \left[\int_c^d D_1 f(x,y)\, dy\right] dx,$$

by Theorem 2. The first term in the last line is constant, so its derivative is zero. The second term is

$$\int_a^t H(x)\, dx, \quad \text{where} \quad H(x) = \int_c^d D_1 f(x,y)\, dy.$$

By Theorem 1, H is continuous, so by the fundamental theorem of calculus $G(t) = \int_a^t H(x)\, dx$ has the derivative

$$G'(t) = H(t) = \int_c^d D_1 f(t,y)\, dy. \quad Q.E.D.$$

Perhaps it is fitting to write Leibniz' rule in Leibniz notation:

$$\frac{d}{dt}\int_c^d f(t,y)\, dy = \int_c^d \frac{\partial f}{\partial t}(t,y)\, dy.$$

Thus, under appropriate conditions, the derivative d/dt can be moved from the outside of the integral sign to the inside. (But once inside, it has to be rewritten as $\partial/\partial t$, since it now applies to a function of two variables.) In this guise, it is easy to see how the rule works when other letters are used for the variables; for example,

$$\frac{d}{dx}\int_a^b f(x,s)\, ds = \int_a^b \frac{\partial f}{\partial x}(x,s)\, ds,$$

$$\frac{d}{dt}\int_a^b f(t,x)\, dx = \int_a^b \frac{\partial f}{\partial t}(t,x)\, dx.$$

Further, since s is held constant in computing $\partial f(t,s,x)/\partial t$, we have

$$\frac{\partial}{\partial t}\int_a^b f(t,s,x)\, dx = \int_a^b \frac{\partial f}{\partial t}(t,s,x)\, dx.$$

Example 6. Let

$$G(t) = \int_0^1 \sin\,(t+y)\,dy.$$

Applying Theorem 3 with $f(t,y) = \sin\,(t+y)$, we find $D_1 f(t,y) = f_t = \cos\,(t+y)$; hence

$$G'(t) = \int_0^1 \cos\,(t+y)\,dy = [\sin\,(t+y)]_0^1$$

$$= \sin\,(t+1) - \sin t.$$

We can check this by first evaluating the integral for $G(t)$, obtaining $G(t) = -\cos\,(t+1) + \cos t$, which yields $G'(t) = \sin\,(t+1) - \sin t$.

Example 7. Let

$$G(t) = \int_1^2 \frac{1}{y}\,e^{ty}\,dy.$$

Applying Theorem 3 with

$$f(t,y) = \frac{1}{y}\,e^{ty},$$

we find $D_1 f(t,y) = f_t = e^{ty}$; hence

$$G'(t) = \int_1^2 e^{ty}\,dy = \left[\frac{1}{t}\,e^{ty}\right]_1^2 = \frac{e^{2t} - e^t}{t}.$$

We cannot check this result by evaluating $G(t)$ explicitly, since none of the standard methods evaluate the integral

$$\int \frac{1}{y}\,e^{ty}\,dy.$$

In a case like this, Theorem 3 is the only practical way to obtain a useful expression for $G'(t)$.

Example 8. Let

$$G(x,y) = \int_0^x f(s,y)\,ds,$$

where f and f_y are continuous. Find G_x and G_y. *Solution.* By the fundamental theorem of calculus, $G_x(x,y) = f(x,y)$, and by Leibniz' rule,

$$G_y(x,y) = \int_0^x f_y(s,y)\,ds.$$

Example 9. Let

$$H(x,y) = \int_0^{\varphi(x,y)} f(s,y)\,ds,$$

where f, f_y, φ, φ_x, and φ_y are all continuous. Find H_x and H_y. *Solution.* We can use the chain rule, with the scheme

$$(x,y) \;\rightarrow\; (\varphi(x,y),y) \;\rightarrow\; \int_0^{\varphi(x,y)} f(s,y)\,ds = H(x,y)$$

or

$$(x,y) \;\rightarrow\; (u,v) \;\rightarrow\; z$$

where $u = \varphi(x,y)$, $v = y$, and $z = \int_0^u f(s,v)\,ds$. We find

$$H_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= f(u,v)\cdot\varphi_x + \left(\int_0^u D_2 f(s,v)\,ds\right)\cdot 0$$

$$= f(\varphi(x,y),y)\varphi_x(x,y),$$

and

$$H_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= f(u,v)\cdot\varphi_y + \left(\int_0^u D_2 f(s,v)\,ds\right)\cdot 1$$

$$= f(\varphi(x,y),y)\varphi_y(x,y) + \int_0^{\varphi(x,y)} D_2 f(s,y)\,ds.$$

PROBLEMS

1. Evaluate the following repeated integrals.

(a) $\displaystyle\int_0^\pi \left[\int_0^y y \sin x\,dx\right] dy$ (d) $\displaystyle\int_0^1 \left[\int_{\sqrt{x}}^1 dy\right] dx$

(b) $\displaystyle\int_1^2 \left[\int_x^{x^2} dy\right] dx$ (e) $\displaystyle\int_0^2 \left[\int_1^{e^y} dx\right] dy$

(c) $\displaystyle\int_1^3 \left[\int_0^{\log y} e^{x+y}\,dx\right] dy$ (f) $\displaystyle\int_0^1 \left[\int_0^{x^2} \frac{1}{1+x^2}\,dy\right] dx.$

2. Describe each of the sets in Fig. 9 in the form (4), and in the form (6).

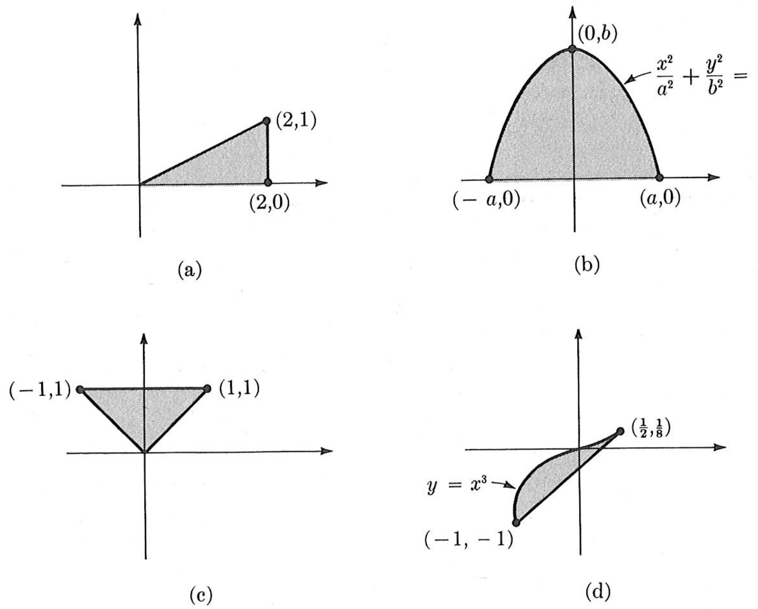


FIGURE 4.9

3. For each of the integrals in Problem 1, sketch the set S over which you are integrating, and rewrite the integral with dx and dy interchanged. If possible, evaluate the new repeated integral, and compare it to the result found in Problem 1. (*Warning:* It is difficult to obtain the limits for the new repeated integral just by looking at the original limits; you need a clear picture of the set S over which you are integrating.)

4. Find the partial derivatives of the following functions by applying the fundamental theorem of calculus and Leibniz' rule.

(a) $f(x,y) = \int_1^x e^{sy}\, ds$

(b) $g(x,y) = \int_1^{x^2} e^{sy}\, ds$

(Hint: $g(x,y) = f(x^2,y)$, with f as in (a); use the chain rule).

(c) $h(x,y) = \int_0^x \sin(sy)\, ds$

(d) $i(x,y) = \int_0^{x^3} \sin(sy)\, ds$

(e) $j(x,y) = \int_0^{x+y} \sin(sy)\, ds$

(Hint: $j(x,y) = h(x+y,y)$; use the chain rule.)

(f) $k(x,y) = \int_0^{\varphi(x)} \sin(sy)\, ds$, where φ is differentiable

(g) $l(x,y) = \int_0^{\varphi(x,y)} \sin(sy)\, ds$, where φ is differentiable

(h) $m(x,y) = \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} \sin(sy)\, ds$ (Hint: $\int_{\varphi_1}^{\varphi_2} = \int_0^{\varphi_2} - \int_0^{\varphi_1}$)

5. Evaluate the integrals in Problem 4(a)–(e), and check the results you found in Problem 4.

6. Suppose that f and f_y are continuous on the rectangle

$$R = \{(x,y) : a \le x \le b,\ c \le y \le d\},$$

that φ_1 and φ_2 are differentiable on R, that $a \le \varphi_1 \le b$, and that $a \le \varphi_2 \le b$. Prove that if

$$F(x,y) = \int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(s,y)\, ds,$$

then

$$F_x(x,y) = f(\varphi_2(x,y),y)\frac{\partial \varphi_2}{\partial x} - f(\varphi_1(x,y),y)\frac{\partial \varphi_1}{\partial x},$$

and

$$F_y(x,y) = \int_{\varphi_1}^{\varphi_2} f_y(s,y)\, ds + f(\varphi_2(x,y),y)\frac{\partial \varphi_2}{\partial y} - f(\varphi_1(x,y),y)\frac{\partial \varphi_1}{\partial y}.$$

(Hint: Do Problem 4 first.)

7. Suppose that S is a region of the type in Theorem 2, and f and g are continuous on S.

(a) Prove that $\iint_S (\alpha f + \beta g) = \alpha \iint_S f + \beta \iint_S g$ for all constants α and β.

(b) Prove that if $f \ge g$, then $\iint_S f \ge \iint_S g$.

8. (a) Suppose that U is an open set, f is continuous on U, and $\iint_D f = 0$ for every disk D contained in U. Prove that $f \equiv 0$ in U. (Hint: If $f \not\equiv 0$, then $f(\mathbf{P}_0) \ne 0$ for some point $\mathbf{P}_0$ in U; say $f(\mathbf{P}_0) > 0$.

Since f is continuous and U is open, there is a disk D contained in U such that $f(\mathbf{P}) > \frac{1}{2}f(\mathbf{P}_0)$ for every point $\mathbf{P}$ in R. Now apply Problem 7(b).)

(b) Suppose that S is a region of the type in Theorem 2, f is continuous on S, $f \geq 0$, and $\iint_S f = 0$. Let U be any open set contained in S. Prove that $f \equiv 0$ on U. (Hint: Show that $0 \leq \iint_D f \leq \iint_S f = 0$ for every disk D contained in U.)

References. The following books give a thorough development of double integrals, including the proofs omitted above. We will refer to them again later in connection with Green's theorem and its extensions.

T. M. Apostol, *Mathematical Analysis*, Addison-Wesley, 1957

W. H. Fleming, *Functions of Several Variables*, Addison-Wesley, 1965

M. Spivak, *Calculus on Manifolds*, Benjamin, 1965

J. W. Woll, Jr., *Functions of Several Variables*, Harcourt Brace, 1966

4.2 VECTOR FIELDS

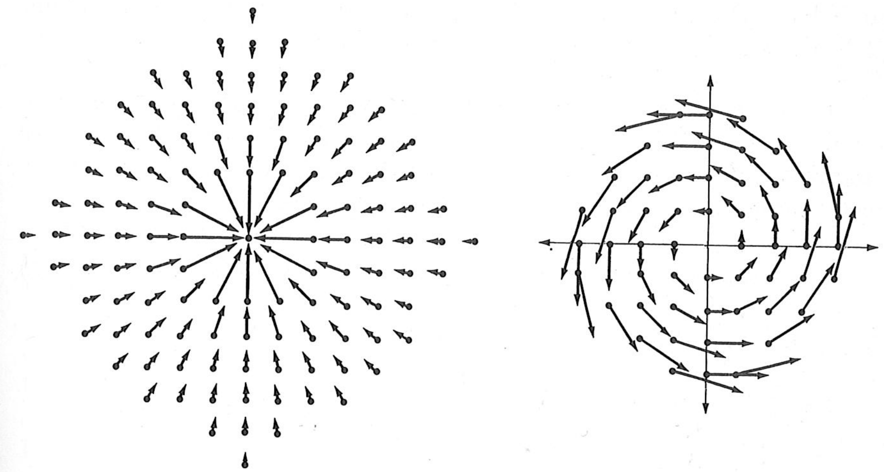
A vector field $\mathbf{F}$ over a set S is a function assigning a vector $\mathbf{F}(\mathbf{P})$ to each point $\mathbf{P}$ in S. Here the set S will be in the plane, and the vectors will be in R^2. The two components of $\mathbf{F}(\mathbf{P})$ are often denoted $M(\mathbf{P})$ and $N(\mathbf{P})$; thus $\mathbf{F}(\mathbf{P}) = (M(\mathbf{P}), N(\mathbf{P}))$, where M and N are ordinary real-valued functions on S. To visualize $\mathbf{F}$, picture at each point $\mathbf{P}$ in S an arrow representing $\mathbf{F}(\mathbf{P})$, as in Fig. 10.

The two main physical examples are *force fields* and *velocity fields*. A particle of mass M at the origin attracts a particle of mass m at the point $\mathbf{P}$ by a force $-\gamma mM\mathbf{P}/|\mathbf{P}|^3$, where γ is a gravity constant. The function

$$\mathbf{F}(\mathbf{P}) = -\gamma mM\,|\mathbf{P}|^{-3}\mathbf{P} \qquad (*)$$

is a vector field defined on the set where $|\mathbf{P}| \neq 0$ (Fig. 10(a)); since it describes a force, it is called a *force field*.

For a simple example of a velocity field, imagine that a plate lying on the plane is rotated counterclockwise about the origin at a rate of c radians per minute. Then for any (x,y) in the plane, the point on the plate lying over (x,y) moves with velocity $\mathbf{F}(x,y) = (-cy,cx)$, as you



(a) $\mathbf{F}(\mathbf{P}) = \dfrac{1}{|\mathbf{P}|^3}\mathbf{P}$ (b) $\mathbf{F}(x,y) = \left(-\dfrac{y}{2}, \dfrac{x}{2}\right)$

FIGURE 4.10

can easily check (see Problem 4). This vector field $\mathbf{F}$ is called the *velocity field* of the rotation (Fig. 10(b)).

In a purely mathematical context, vector fields arise from gradients; precisely, if f is differentiable at every point in S, then $\mathbf{F}(\mathbf{P}) = \nabla f(\mathbf{P})$ defines a vector field $\mathbf{F}$ on S. For example, if $f(\mathbf{P}) = \gamma mM/|\mathbf{P}|$, you can check that $\nabla f(\mathbf{P}) = -\gamma mM\,|\mathbf{P}|^{-3}\,\mathbf{P}$, and we get the force field (*) as a gradient. The function f is called a *potential function* of the force $\mathbf{F} = \nabla f$.

Not every vector field is a gradient, though. For example, the velocity field $\mathbf{F}(x,y) = (-cy,cx)$ is not the gradient of any function f. Suppose, on the contrary, that $\mathbf{F}$ were a gradient ∇f; then we would have $f_x = -cy$, $f_y = cx$; hence $f_{xy} = -c$, $f_{yx} = c$, and the mixed partials of f would be continuous but unequal, contradicting Theorem 10 of the previous chapter. Hence there is no function f such that $\nabla f(x,y) = (-cy,cx)$.

This raises the question: Which vector fields are gradients? Or, in physical terms, which force fields have potentials? The answer is easy for a vector field defined over a rectangle:

Theorem 4. *Suppose that* $\mathbf{F} = (M,N)$ *is defined on an open rectangle*

$$R = \{(x,y) : a < x < b,\ c < y < d\},$$

and the partial derivatives M_y *and* N_x *are continuous on* R. *Then* $\mathbf{F}$ *is a gradient if and only if* $M_y \equiv N_x$.