MA 510 first midterm review problems

Hopefully final version as of February 13th.

The first midterm will be in class on Wednesday, February 19th. It will cover all the material from the classes and homework up to that date. Notes, books, and electronic devices are not allowed, but there is a reference page at the back which will also be included on the actual exam. Most of the problems on the exam will be closely based on ones from the list below. For each problem, you must justify your answers. Please let me know if you have a question or find a mistake.

1. Let $(a, b, 0)$, $(0, a, b)$, and $(a, 0, b)$ be three (not necessarily distinct) points in $\mathbb{R}^3$, where a and b are real numbers.
   (a) For which real values of a and b are the points collinear?
   (b) For which values are they coplanar with the origin?
   (c) How do the answers change if the third point is replaced by $(b, 0, a)$?

2. (a) Find a normal unit vector to the plane in $\mathbb{R}^3$ given by $x - 2y - 2z = 1$.
   (b) Find the intersection of this plane with the line perpendicular to it and passing through $(1, 2, 3)$.

3. What is the cosine of the angle between the diagonal of a cube and one of the edges? What about between the diagonal of a cube and the diagonal of one of its sides?

4. Find the equation for the surface consisting of those points in $\mathbb{R}^3$ which are equidistant from the y-axis and the xz-plane. Sketch and describe the surface.

5. A train track is to be built up a hill given by the graph in $\mathbb{R}^3$ of the function $f(x, y) = 10 - x^2 - 2y^2$. What are the possible directions the track can head starting at $(1, 0)$ so that it is going uphill at a slope of $1/2$?

6. Let $z = f(2x + 3y, 2x - 3y)$, where $f(u, v) = \sin(\cos(\exp(1 + \ln(1 + u^2 + v^2))))$. Find constants a and b such that
$$\partial_x z \partial_y z = a(\partial_u f)^2 + b(\partial_v f)^2.$$

7. Let $u = (385736, -427850, 1249)$, let $v = 4u$ and let $w = -3u/\|u\|^2$. Find $v \cdot w$ and $v \times w$.

8. Let $f(x, y) = \ln(2x^2 - y)$. Find the linear approximation to f at $(1, 1)$.

9. Let $v = (a, 1)$, where a is a given real number. Let $u = v/\|v\|$. Let $f(x, y)$ be a

given differentiable function such that $\nabla f(1,2)$ points towards the origin. Let b be the directional derivative of f in the direction u. For which values of a is b positive, for which is it negative, and for which is it 0?

10. Let

$$\begin{aligned} f(x_1, x_2) &= (x_1 x_2 - x_1), (x_1 - 1)^4) \\ g(y_1, y_2) &= (y_1^2, e^{y_1} + e^{-y_2}) \\ h(z_1, z_2) &= (\sin z_1 + \cos z_2, \cos z_1 - \sin z_2), \end{aligned}$$

and let $F(z_1, z_2) = f(g(h(z_1, z_2)))$. Find $DF(0,0)$.

11. Suppose that $f \colon \mathbb{R}^3 \to \mathbb{R}$ is differentiable and satisfies

$$\nabla f(1, -e, e) = (1, 2, 3),$$

and that $z = z(x,y)$ is given by the implicit equation

$$x - y + z + \ln z = 2 + 2e,$$

where x and y are independent variables. If $g(x,y) = f(x, y, z(x,y))$, what is $\nabla g(1, -e)$?

12. Let $f = f(u, v, w)$ be a differentiable function $\mathbb{R}^3 \to \mathbb{R}$, and let

$$g(x,y) = f(x-y, x+y, xy)$$

   (a) If $(4, 10, 21)$ is a critical point of f, which point must be a critical point of g?

   (b) Calculate $\partial_x \partial_y g$ in terms of the derivatives of f.

13. Let a be a real number, and let $f(x,y,z) = x^2 + y^2 + z^2 + ayz$. For each value of a, decide if the critical point at the origin is a local maximum, a local minimum, or a saddle point.

14. Find the largest possible value of $f(x,y,z) = xy^2z^3$ such that $x^2 + 4y^2 + 9z^2 = 1$.

15. Let C be the curve given by the intersection of the cylinder $x^2 + y^2 = 4$ and the plane $x + y + z = 1$. Sketch C. Find a constant a such that the arclength of C is given by

$$a \int_0^{2\pi} \sqrt{1 - \cos t \sin t} dt.$$

Evaluate $\int_C (4 - xy)^{-1/2} ds$.

Reference page

$u \cdot v = \|u\|\|v\| \cos\theta$, $\|u \times v\| = \|u\|\|v\||\sin\theta|$, where θ is the angle between u and v.

The unit vector in the direction of v is $u = v/\|v\|$. The directional derivative is then $\nabla f \cdot u$.

The eigenvalues of a matrix A are the solutions to $\det(A - \lambda I) = 0$. If the matrix of second partial derivatives of f has all positive eigenvalues at a critical point then f has a local minimum there.

The arclength of a curve parametrized by $c(t)$ from $c(a)$ to $c(b)$ is $\int_a^b \|c'(t)\| dt$.