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MA 530
Spring 2023

Homework 4

Due February 22nd on paper at the beginning of class. Please let me know if you have a question or find a mistake.

- Section 2.6: # 1, # 1A, # 2.
- Section 2.7: # 3.
- Section 2.8: # 4, # 5.
- Additional Problem: Let m_0, m_1, ... be a strictly increasing sequence of nonnegative integers and let $f(z) = \sum_{n=0}^{\infty} z^{m_n}$.
  1. Prove that, for any such choice of m_n's, $f(z)$ is a holomorphic function on the unit disk $D = \{z \in \mathbb{C} \colon |z| < 1\}$.
  2. Suppose there is a positive integer k such that $m_n = kn$ for every n. Find points $\omega_{k,1}, \ldots, \omega_{k,k} \in \partial D$ such that $f(z)$ continues analytically from D to $\mathbb{C} \setminus \{\omega_{k,1}, \ldots, \omega_{k,k}\}$.
  3. Suppose that $m_n = n!$. Prove that, for any $\omega_{k,j}$ as in part 2 above, we have

  $$\lim_{r \to 1^-} |f(r\omega_{k,j})| = \infty.$$

  Hints: For part 1, compare $\sum_{n=0}^{\infty} z^{m_n}$ with the geometric series $\sum_{n=0}^{\infty} |z|^n$. For part 2, observe that the series $\sum_{n=0}^{\infty} z^{m_n}$ is geometric in this case and use that to simplify f. For part 3, first show that for any $\omega_{k,j}$, there is an N such that $(r\omega_{k,j})^{n!} = r^{n!}$ when $n \geq N$. Next show that $\operatorname{Re} f(r\omega_{k,j}) > \sum_{n=0}^{M} \operatorname{Re}(r\omega_{k,j})^{n!}$ for any $M > N$. Next show that, for any postive A, there exist $M > N$ and $R \in (0, 1)$, such that $\sum_{n=0}^{M} \operatorname{Re}(r\omega_{k,j})^{n!} > A$ when $r \in (R, 1)$. Finally, use this to conclude that $\lim_{r \to 1^-} \operatorname{Re} f(r\omega_{k,j}) = \infty$.

  Interpretation: In part 2, $f(z)$ is holomorphic up to and beyond the boundary of its disk of convergence, except for finitely many points where it has poles. In part 3, f blows up on a dense subset of this boundary, and hence it follows that the whole boundary of the disk of convergence is a wall of singularities for f.

  Further reading: For more on such series, see `https://en.wikipedia.org/wiki/Lacunary_function`. A related but more complicated situation is discussed in a beautiful lecture of Hardy `https://hdl.handle.net/2027/uc2.ark:/13960/t1zc7tj66` (try not to be discouraged by how often he calls something 'trivial' even when it is quite difficult for most people!) The function with finitely many poles is on page 10 there, and the wall of singularities is on page 13.