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MA 530
Spring 2023

Homework 8

Due April 12th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. Let $\zeta(s)$ be the Riemann zeta function.
   (a) Prove that $\zeta(s) = \overline{\zeta(\bar{s})}$ for all $s \in \mathbb{C} \setminus \{1\}$

   Hint: This is a variant of Exercise 1 from Section 2.6 of Taylor. Use Equation (4.4.1).

   (b) Use the result of part (a) to prove that the zeroes of ζ are arranged symmetrically with respect to the real axis (i.e. the complex conjugate of a zero is a zero).

   (c) Use Equation (4.4.18) from Taylor to prove that the zeroes of ζ in the critical strip $\{s \in \mathbb{C}\colon 0 < \operatorname{Re} s < 1\}$ are arranged symmetrically with respect to the critical line $\operatorname{Re} s = 1/2$.

   (d) Use Equation (4.4.18) from Taylor to evaluate $\zeta(0)$ and $\zeta(-1)$.

   Interpretation: Thus, writing $\zeta(-k) = 1^k + 2^k + 3^k + \cdots$, the ζ function *regularizes* the divergent sums $1+1+1+\cdots$ and $1+2+3+\cdots$ to yield a finite answer rather than $+\infty$. This remarkable method is used to calculate energies and forces in quantum mechanics, with ζ making an appearance for the Casimir force between two parallel plates (see page 26 of `https://arxiv.org/pdf/quant-ph/0106045.pdf`). A simpler version of the same method, using $1/(1-z) = 1 + z + z^2 + \cdots$, gives $1 + 2 + 4 + 8 + \cdots = -1$. In this case convergence also holds with respect to the 2-adic metric, which is important in number theory, and can be used to solve Diophantine equations (see the first lecture here `https://sites.math.rutgers.edu/~alexk/2023S572/index.html`).

2. Let $\Omega = \{z \in \mathbb{C}\colon \operatorname{Re} z > 0\}$ and let $D = \{z \in \mathbb{C}\colon |z| < 1\}$.
   (a) Find a biholomorphism $g\colon \Omega \to D$ such that $g(2023) = 0$.

   (b) Find $R > 0$ such that, for any $q \in D$, the following are equivalent:
      i. $|q| = R$.
      ii. There exists a biholomorphism $g\colon \Omega \to D$ obeying $g(1) = 0$ and $g(1/2) = q$.

   Hint: Use the results of Exercise 12 from Section 1.1 of Taylor and Example 4 from Section 3.4 of Fisher.