MA 562 final exam review problems
Version as of December 6th.

The final will be on Tuesday, December 13th, from 1 to 3 pm in UNIV 101. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Let M consist of the points in $\mathbb{R}^4$ which are equidistant from the plane $x^2 = 1$ and the x^1 axis. Prove that M is a regular submanifold of $\mathbb{R}^5$. What is its dimension?

2. Let N be a closed regular submanifold of a manifold M. Let $f \in C^\infty(N)$ be bounded. Show that there is a function $h \in C^\infty(M)$ such that $f(p) = h(p)$ for all $p \in N$ and such that $|h(p)| \leq \sup |f|$ for all $p \in M \setminus N$.

3. Let
$$A = \begin{bmatrix} 0 & 1 & -2 & 3 & -4 \\ -1 & 0 & 5 & -6 & 7 \\ 2 & -5 & 0 & -8 & 9 \\ -3 & 6 & 8 & 0 & -10 \\ 4 & -7 & -9 & 10 & 0 \end{bmatrix}$$
and consider the vector field on $\mathbb{R}^5$ given by $V(x) = Ax$. Let $\theta(t,p)$ be the corresponding one-parameter group, $\theta \colon \mathbb{R} \times \mathbb{R}^5 \to \mathbb{R}^5$.
   (a) Prove that, for every $p \in \mathbb{R}^5$, the distance from the origin of $\theta(t,p)$ is independent of t.
   (b) Prove that, for every $p \in \mathbb{R}^5$ and every u in the kernel of A, the dot product $\theta(t,p) \cdot u$ is independent of t.
   (c) Conclude that, for every $p \in \mathbb{R}^5$, there is a 3-sphere S^* such that the orbit $\{\theta(t,p) \colon t \in \mathbb{R}\}$ is contained in S^*

4. Let $M = \{(w,x,y,z) \in \mathbb{R}^4 \colon 2w^2 + 3x^2 + 4y^2 + 5z^2 = 6\}$. Show that M is an orientable smooth manifold, specify an orientation on M, and find the volume element corresponding to that orientation and to the Riemannian metric inherited from $\mathbb{R}^4$.

5. Let a, b, and c be given positive numbers, and consider the differential form
$$\omega = \frac{xdy \wedge dz + ydz \wedge dx + zdx \wedge dy}{(ax^2 + by^2 + cz^2)^{3/2}}$$
on $\mathbb{R}^3 \setminus \{(0,0,0)\}$. Is ω closed? Is it exact?

   Hint: Use a rescaling change of coordinates to pull out the constants.

6. Let $\alpha = Ax\,dx + By\,dy$ and $\beta = Cy\,dx + Dx\,dy$, where A, B, C, D are given real numbers. Find all 1-forms ω on $\mathbb{R}^2 \setminus \{0\}$ such that
$$d\alpha = \omega \wedge \beta,$$
$$d\beta = \alpha \wedge \omega.$$

   Hint: Apply both sides to (∂_x, ∂_y).

7. Let $a \neq 0$ be given and let $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\varphi(u, v) = (u \cos v, u \sin v, av)$. Prove that φ is an immersion (can you show it's an embedding? the resulting surface is a helicoid, something we've looked at before), and find functions f and g such that $f(u, v)du^2 + g(u, v)dv^2 = \varphi^*\Phi_E$, where Φ_E is the standard Riemannian metric on $\mathbb{R}^3$. Find functions h_1 and h_2 such that $e_1 = h_1\partial_u$ and $e_2 = h_2\partial_v$ are an orthonormal frame, and find the corresponding dual frame ω_1, ω_2. Use the structure equations $d\omega_j = \sum_k \omega_{jk} \wedge \omega_k$ to find the connection form ω_{12}, and the formula $d\omega_{12} = -K\omega_1 \wedge \omega_2$ to find the Gaussian curvature K.

8. Let $f\colon \mathbb{R} \to \mathbb{R}$ be C^∞. Define a vector field X on $\mathbb{R}^2$ by $X(x, y) = \partial_x + f'(x)\partial_y$. Find the one-parameter group generated by X. Let $e_1 = X/|X|$, and let e_2 be the vector orthonormal to e_1 and in the standard orientation on $\mathbb{R}^2$. Find e_1, e_2 and the dual vectors ω_1, ω_2 in terms of f, ∂_x, ∂_y, dx, dy. Use the structure equations $d\omega_j = \sum_k \omega_{jk} \wedge \omega_k$ to find ω_{12} and $\omega_{12}(e_1)$ at all (x, y) such that $f'(x) = 0$. (Incidentally $\omega_{12}(e_1)$ at a point (a, b) is the geodesic curvature of the integral curve of X at the point (a, b))