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MA 562
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Homework 1

Due September 2nd on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. Consider the two-sphere $\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$.
   (a) Let $U_1 = \mathbb{S}^2 \cap \{z > 0\}$ and $\psi_1(x,y,z) = (x,y)$ be a coordinate neighborhood. Find $\psi_1(U_1)$ and compute $\psi_1^{-1} \colon \psi_1(U_1) \to \mathbb{S}^2$.
   (b) Repeat part (a) for $U_2 = \mathbb{S}^2 \cap \{y < 0\}$ and $\psi_2(x,y,z) = (x,z)$.
   (c) Find $\psi_1(U_1 \cap U_2)$ and $\psi_2(U_1 \cap U_2)$ and compute $\psi_2 \circ \psi_1^{-1}$ and $\psi_1 \circ \psi_2^{-1}$.
   (d) Let U_3 be the range of $(0,2\pi) \times (0,\pi) \ni (\theta,\varphi) \mapsto (\cos\theta\sin\varphi, \sin\theta\sin\varphi, \cos\varphi)$ and ψ_3 the inverse of this map. Repeat part (c) with U_3 and ψ_3 replacing U_2 and ψ_2.

2. Let M be the the closed unit disk $\{(x,y) \in \mathbb{R}^2 \mid r \le 1\}$ with opposite boundary points identified, where $r = \sqrt{x^2+y^2}$. Prove that

$$\begin{aligned}
U_1 &= \{r < 1\}, & \psi_1(x,y) &= (x,y),\\
U_2 &= M \cap \{x \neq 0\}, & \psi_2(x,y) &= (\arctan(y/x), (1-r)\operatorname{sgn} x),\\
U_3 &= M \cap \{y \neq 0\}, & \psi_3(x,y) &= (\arctan(x/y), (1-r)\operatorname{sgn} y).
\end{aligned}$$

   is an atlas on M, and compute all the transition functions, including their domains and ranges.

3. Find a function g such that $(x,y) \mapsto [x,y,g(x,y)]$ is a bijection from the manifold of Problem 2 to $P^2(\mathbb{R})$. You don't have to check this explicitly, but your final result should be a homeomorphism between M and $P^2(\mathbb{R})$, and maybe even a diffeomorphism. Thus we have another picture of the projective plane as a manifold.