Kiril Datchev
MA 562
Fall 2022

Homework 2

Due September 9th on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. Exercise II.2.2.

2. Exercise II.2.5 (second or revised second edition), aka Exercise II.2.4 (first edition).

3. Let M be a set, and let m and n be positive integers. Let U_α and U_β be subsets of M with $U_\alpha \cap U_\beta \neq \varnothing$. Let $\psi_\alpha \colon U_\alpha \to \mathbb{R}^n$ and $\psi_\beta \colon U_\beta \to \mathbb{R}^m$ be one-to-one functions such that the sets $\psi_\alpha(U_\alpha \cap U_\beta)$ and $\psi_\beta(U_\alpha \cap U_\beta)$ are open. Suppose the functions $\psi_\alpha \circ \psi_\beta^{-1} \colon \psi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ and $\psi_\beta \circ \psi_\alpha^{-1} \colon \psi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^m$ are C^1. Use the result of the previous exercise to show that $n = m$.

4. Exercise II.5.5. Where it says "$Y = \sum \beta^i(x)\partial/\partial y^i \varepsilon \mathfrak{X}(U)$", it should be "$Y = \sum \beta^i(x)\partial/\partial x^i \in \mathfrak{X}(U)$", where $\mathfrak{X}(U)$ is the space of C^∞ vector fields on U.

5. Exercise II.6.3.

6. The reduced version of the inverse function theorem says the following. Let f be a C^1 function mapping a neighborhood of the origin in $\mathbb{R}^n$ to a neighborhood of the origin in $\mathbb{R}^n$, with $f(0) = 0$ and $Df(0) = I$. Then there exist a neighborhood U of the origin in $\mathbb{R}^n$ and a C^1 function $g \colon f(U) \to U$ such that $g(f(x)) = x$ for every $x \in U$ and $f(g(y)) = y$ for every $y \in f(U)$.

   The more general version of the inverse function theorem says the following. Let $a \in \mathbb{R}^n$, and let F be a C^1 function mapping a neighborhood of a to $\mathbb{R}^n$, with $DF(a)$ invertible. Then there exist a neighborhood W of a and a C^1 function $G \colon F(W) \to W$ such that $G(F(x)) = x$ for every $x \in U$ and $F(G(y)) = y$ for every $y \in F(U)$.

   Deduce the more general version from the reduced version by composing with strategically chosen mappings of the kind given in Examples II.6.1 and II.6.2.