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MA 562
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Homework 9

Due December 2nd on paper at the beginning of class. Please let me know if you have a question or find a mistake.

1. Exercise 14 from Chapter 4, page 74 of do Carmo's book `https://link-springer-com.ezproxy.lib.purdue.edu/book/10.1007/978-3-642-57951-6`. For part (b), give two constructions of a suitable triple (P, Q, R) as follows. Give one following the hint: this is an implementation of a general formula from Poincaré's lemma, and is applicable to differential forms of any degree in any dimension. Give a second, simpler one, which works in this special case by setting one of P, Q and R equal to zero and then solving the system by integration and back substitution. Prove the difference $(P_1, Q_1, R_1) - (P_2, Q_2, R_2)$ between the two solutions is given by $(\partial_x f, \partial_y f, \partial_z f)$ for some smooth f, and find f in terms of $P_1,\ P_2,\ Q_1,\ Q_2,\ R_1,\ R_2$.

2. Let $\varphi(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ be the cylindrical coordinate map. Find functions a, b, and c, such that $e_1 = \varphi_* a\partial r$, $e_2 = \varphi_* b\partial\theta$ and $e_3 = \varphi_* c\partial z$ are an orthonormal frame. Compute the pull-backs of the dual frame $\varphi^*\omega_j$ for $j = 1, 2, 3$ and of the connection one forms $\varphi^*\omega_{jk}$ for $j,\ k = 1, 2, 3$.

3. (a) Exercise 5 from Chapter 5, page 97 of do Carmo's book `https://link-springer-com.ezproxy.lib.purdue.edu/book/10.1007/978-3-642-57951-6`. To solve this, find an orthonormal frame of the form $e_1 = a\partial_x$, $e_2 = b\partial_y$, for suitable positive functions a and b. Then find the dual frame ω_1, ω_2, and find ω_{12} by writing $\omega_{12} = f dx + h dy$ and solving for f and h by plugging into $d\omega_1 = \omega_{12} \wedge \omega_2$ and $d\omega_2 = \omega_1 \wedge \omega_{12}$. Finally solve for K by plugging into $d\omega_{12} = -K\omega_1 \wedge \omega_2$.

   (b) The most important case is when K is a constant. Let $U = \{(x, y) \colon y > 0\}$. For each constant $c < 0$, find $g \colon U \to (0, \infty)$ such that $K(x, y) = c$ for all $(x, y) \in U$. (This generalizes Exercise 2 from the previous page).