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MA 562
Fall 2022

This note presents the main part of the inverse function theorem, following Theorem II.6.4 of Boothby's *Introduction to Differentiable Manifolds and Riemannian Geometry* and Theorem 1.1.7 of Hörmander's *Analysis of Linear Partial Differential Operators: Volume I*. Questions, comments, and corrections are gratefully received at kdatchev@purdue.edu.

Theorem. *Let $W \subset \mathbb{R}^n$ be an open set and $F\colon W \to \mathbb{R}^n$ a C^ℓ function for some $\ell \geq 1$. Suppose $DF(a)$ is invertible for some $a \in W$. Then there exists a neighborhood U of a such that $F\colon U \to F(U)$ is a C^ℓ diffeomorphism.*

Proof. Our proof is based on a contraction mapping argument. It is a general fact (see Theorem II.6.5 of Boothby) that if $T\colon X \to X$ where X is a complete metric space, and if there is $\lambda \in (0,1)$ such that $d(T(x),T(y)) \leq \lambda d(x,y)$ for all x and y, then the equation $x = T(x)$ has a unique solution and for any x_0 the sequence $x_0, T(x_0), T(T(x_0)), \dots$ converges to it. Our proof does not quote this fact, but along the way proves a special case of it with $\lambda = \frac{1}{2}$ and X a closed ball in $\mathbb{R}^n$.

Step 1: Reduce to the case $a = 0$, $F(0) = 0$, $DF(0) = I$ by composing with translations and linear transformations. This is left as an exercise: see also Examples II.6.1 and II.6.2, and Lemma II.6.3, of Boothby.

Step 2: In this step we find a neighborhood U of 0 such that $F\colon U \to F(U)$ is a bijection. Take y with $|y| \leq r$ for some $r > 0$ to be determined, and define

$$x_0 = 0, \qquad x_k = y + x_{k-1} - F(x_{k-1}).$$

Our goal is to choose r small enough that this sequence converges, because then the limit x^* will obey $x^* = y + x^* - F(x^*)$ and hence solve $F(x^*) = y$. It makes sense that this should be possible because, if $|x|$ is small enough, then the conditions $F(0) = 0$ and $DF(0) = I$ tell us that, roughly speaking, $F(x) \approx x$ and so

$$x_k - x_{k-1} = x_{k-1} - F(x_{k-1}) - x_{k-2} + F(x_{k-2}) \approx 0$$

and we are in the setting of a contraction mapping. More precisely, we will arrange[1] $r > 0$ such that

$$|x|, |\tilde{x}| \leq 2r \quad \Longrightarrow \quad |x - F(x) - \tilde{x} + F(\tilde{x})| \leq \tfrac{1}{2}|x - \tilde{x}|. \tag{1}$$

Assume for the moment that there exists $r > 0$ such that (1) holds. Then, since $|y| \leq r$, we have $|x_1| \leq r$ and

$$|x_k - x_{k-1}| = |x_{k-1} - F(x_{k-1}) - x_{k-2} + F(x_{k-2})| \leq \tfrac{1}{2}|x_{k-1} - x_{k-2}|, \tag{2}$$

for all $k \geq 2$, and hence $|x_k| \leq r + \frac{1}{2}r + \cdots + 2^{1-k}r \leq 2r$ for all k. Thus, using (2) repeatedly,

$$\sum_{k=1}^{\infty} |x_k - x_{k-1}| \leq \sum_{k=1}^{\infty} 2^{1-k}|x_1 - x_0|$$

[1] The use of $\frac{1}{2}$ in the right hand side of (1) corresponds to taking $\lambda = \frac{1}{2}$ in the contraction mapping argument. One could also replace this $\frac{1}{2}$ by any number $\lambda \in (0,1)$, provided one accordingly adjusted the requirement $|x|, |\tilde{x}| \leq 2r$. If Step 2 appears mysterious, working that out might be a helpful exercise.

converges, and hence we may define

$$x^* = \lim_{m\to\infty} x_m = \lim_{m\to\infty} \sum_{k=1}^{m} (x_k - x_{k-1}).$$

Next, x^* solves $F(x^*) = y$, and the solution is unique because for any $\tilde{x}$ with $|\tilde{x}| \le 2r$ which solves $F(\tilde{x}) = y$, by (1) we have $|x^* - \tilde{x}| \le \frac{1}{2}|x^* - \tilde{x}|$. So to complete Step 2, it remains to show that there is $r > 0$ such that (1) holds.

To do that, recall the mean value bound

$$|x|, |\tilde{x}| \le 2r \quad \Longrightarrow \quad |\psi(x) - \psi(\tilde{x})| \le Kn|x - \tilde{x}|,$$

where K is the maximum of $|\partial_i \psi^j(x)|$ for $|x| \le 2r$ and $1 \le i, j \le n$; see Boothby Theorem 2.2. We apply this with $\psi(x) = x - F(x)$, and note that $\partial_i \psi^j(0) = 0$ for all i and j. Thus, since the $\partial_i \psi^j$ are continuous, if r is small enough we have $Kn \le \frac{1}{2}$, which implies (1).

Step 3. We now show the inverse map obtained in Step 2 is differentiable for r small enough. Let r be as above, and if necessary shrink r so that $DF(x)$ is invertible when $|x| \le 2r$: this can be done because $DF(0) = I$ and the determinant function is continuous. Write $x = G(y)$ for $|y| < r$. For $|k| < r - |y|$, put $h = G(y + k) - x$. By definition,

$$F(x + h) - F(x) = DF(x)h + R(x, h)|h|,$$

where $R(x, h) \to 0$ as $|h| \to 0$. Substituting $F(x) = y$, $F(x + h) = y + k$, and $h = G(y + k) - G(y)$, and solving for $G(y + k) - G(y)$, gives

$$G(y + k) - G(y) = DF(x)^{-1}k - DF(x)^{-1}R(x, h)|h|.$$

Thus to prove that G is differentiable, it remains to show that $DF(x)^{-1}R(x, h)|h|/|k| \to 0$ as $|k| \to 0$.

For this we observe that applying (1) with $\tilde{x} = x + h$ gives $|h - k| \le \frac{1}{2}|h|$, and hence $\frac{1}{2}|h| \le |k| \le \frac{3}{2}|h|$. That implies

$$\lim_{|k|\to 0} |DF(x)^{-1}R(x, h)||h|/|k| \le 2 \lim_{|k|\to 0} |DF(x)^{-1}R(x, h)| = 2 \lim_{|h|\to 0} |DF(x)^{-1}R(x, h)| = 0.$$

Step 4. To show that G is C^ℓ, we show that DG is $C^{\ell-1}$. This in turn follows from the fact that DF is $C^{\ell-1}$, together with Cramer's formula: recall that, for any invertible matrix M, Cramer's formula expresses the entries of M^{-1} as rational functions of the entries of M, with nonvanishing denominators. □