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This note presents basic definitions and examples concerning manifolds in a way that avoids mentioning general topology, following the approach of Section 33 of Arnold's *Ordinary Differential Equations.*

The notation and usage are set up to be as consistent as possible with Boothby's *Introduction to Differentiable Manifolds and Riemannian Geometry.* Compare with Definitions I.3.1 and III.1.2 there. Please email any comments or corrections to kdatchev@purdue.edu.

Definitions

Let M be a set. A *coordinate neighborhood* is a subset $U \subset M$ together with a one-to-one function $\psi : U \to \mathbb{R}^n$ such that $\psi(U)$ is open.

Two coordinate neighborhoods $\psi_\alpha \colon U_\alpha \to \mathbb{R}^n$ and $\psi_\beta \colon U_\beta \to \mathbb{R}^n$ are C^0 *consistent* or C^0 *compatible* if either $U_\alpha \cap U_\beta = \varnothing$ or if the following two conditions hold:

1. The sets $\psi_\alpha(U_\alpha \cap U_\beta)$ and $\psi_\beta(U_\alpha \cap U_\beta)$ are open;
2. The functions $\psi_\alpha \circ \psi_\beta^{-1} \colon \psi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ and $\psi_\beta \circ \psi_\alpha^{-1} \colon \psi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ are C^0.

A family $\mathcal{U} = \{(U_\alpha,\ \psi_\alpha) \mid \alpha \in \mathcal{A}\}$ of C^0 consistent coordinate neighborhoods makes up a C^0 *atlas* if $M = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, i.e. if between them the coordinate neighborhoods cover M.

Two C^0 atlases on M are C^0 *equivalent* if their union is an atlas, i.e. if all the coordinate neighborhoods are consistent.

A C^0 atlas $\mathcal{U}$ on M is *maximal*, or *complete*, if any atlas equivalent to $\mathcal{U}$ is contained in $\mathcal{U}$.

A C^0 *manifold structure* is introduced on the set M if a maximal C^0 atlas is prescribed. By Theorem III.1.3 of Boothby[1] it is enough to prescribe a non-maximal atlas.

A C^0 *manifold* is a set M together with a C^0 manifold structure on it.

A subset $V \subset M$ is *open* if $\psi(V \cap U)$ is open in $\mathbb{R}^n$ for every coordinate neighborhood (ψ, U).

If the number n above is the same for every coordinate neighborhood (and this turns out to always be the case for connected manifolds) then it is called the *dimension* of the manifold.

At this point, two further conditions on M are imposed to rule out certain pathological examples. The *Hausdorff* condition says that if x and y are any two distinct points on M, then there are

[1] or Lemma 1.35 of Lee's *Introduction to Smooth Manifolds*, available electronically from the Purdue Library here: `https://purdue.primo.exlibrisgroup.com/permalink/01PURDUE_PUWL/ufs51j/alma99169167029201081`

disjoint open sets V and W in M such that $x \in V$ and $y \in W$. The *second countability* condition says that, if the atlas $\mathcal{U}$ is not already countable, then some atlas equivalent to it is countable.[2]

We define smoother manifolds by replacing C^0 everywhere by C^k for some $k \geq 1$, or $k = \infty$. One can similarly consider real-analytic or complex-analytic manifolds. The regularity of the manifold is given by the regularity of the *transition functions* $\psi_\alpha \circ \psi_\beta^{-1}$. Another term for C^0 manifold is *topological manifold*, and another term for C^∞ manifold is *smooth manifold*.

Let M and $\tilde{M}$ be two manifolds. A function $f\colon M \to \tilde{M}$ is C^0 provided $\tilde{\psi} \circ f \circ \psi^{-1} \colon U \to \tilde{U}$ is C^0 for any coordinate neighborhoods (U, ψ) and $(\tilde{U}, \tilde{\psi})$ on M and $\tilde{M}$. The function is a *homeomorphism* if it is invertible with C^0 inverse. Similarly, we can define C^k functions provided both manifolds are C^k. If a function and its inverse are both C^k, the function is a C^k *diffeomorphism*, and a C^∞ diffeomorphism is simply a *diffeomorphism*.

Examples

1. Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$. Define an atlas with two coordinate neighborhoods as follows. Let U_1 consist of the points in M with $x < 1$, and put $\psi_1(x, y, z) = (\arg(x + iy), z)$. Let U_2 consist of the points in M with $x > 0$, and put $\psi_2(x, y, z) = (y, z)$. Then $\psi_2(\psi_1^{-1}((\theta, z))) = (\sin\theta, z)$, $\psi_1(\psi_2^{-1}((y, z))) = (\sin^{-1} y, z)$ for $y > 0$, and $\psi_1(\psi_2^{-1}((y, z))) = (\sin^{-1} y + 2\pi, z)$ for $y < 0$.

2. Define *real projective space* $P^n(\mathbb{R})$ to be $\mathbb{R}^{n+1} - \{0\}$ subject to the equivalence relation $x \sim y$ when $y = tx$ for some real number t. To define coordinate neighborhoods, for each point in $P^n(\mathbb{R})$, pick a representative $(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} - \{0\}$. The definitions which follow will be independent of the representative chosen. For each $j \in \{1, \dots, n+1\}$, let $U_j = P^n(\mathbb{R}) \cap \{x_j \neq 0\}$, and let

$$\begin{aligned} \psi_1(x_1, \dots, x_{n+1}) &= (x_2, \dots, x_{n+1})/x_1, \\ \psi_2(x_1, \dots, x_{n+1}) &= (x_1, x_3, \dots, x_{n+1})/x_2, \\ &\vdots \\ \psi_{n+1}(x_1, \dots, x_{n+1}) &= (x_1, x_2, \dots, x_n)/x_{n+1}. \end{aligned}$$

We abbreviate the above by writing $\psi_j(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_j, \dots, x_{n+1})$.

Fix j and k in $\{1, \dots, n+1\}$ with $j < k$. Then $\psi_k(U_j \cap U_k) = \{(y_1, \dots, y_n) \in \mathbb{R}^n \mid y_j \neq 0\}$. A representative of $\psi_k^{-1}(y_1, \dots, y_n)$ is $(y_1, \dots, y_{k-1}, 1, y_k, \dots y_n)$, and so

$$\psi_j(\psi_k^{-1}(y_1, \dots y_n)) = (y_1, \dots, \hat{y}_j, \dots, y_{k-1}, 1, y_k, \dots, y_n)/y_j.$$

This is a continuous map from $\psi_k(U_j \cap U_k)$ to $\mathbb{R}^n$. A similar proof shows that $\psi_k \circ \psi_j^{-1}$ is continuous. Thus the atlas $\{(U_j, \psi_j) \mid j \in \{1, \dots, n+1\}\}$ defines a C^0 manifold structure on $P^n(\mathbb{R})$.

[2] It follows that any atlas has a countable subfamily which is also an atlas. To see this, first observe that having a countable atlas implies that there is a countable basis of open sets, e.g. the preimages under the ψ's of open rectangles with rational coordinates, and second that hence every open cover has a countable subcover (aka every second countable space is Lindelöf: see Theorem 30.3 of Munkres' *Topology* or `https://math.stackexchange.com/questions/1742638/choice-of-chart-and-atlas` and `https://topospaces.subwiki.org/wiki/Second-countable_implies_Lindelof`.