MA 562 first midterm review problems

Version as of September 22nd.

The first midterm will be in class on Monday, September 26th. No notes or electronic devices allowed. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Let M and N be smooth manifolds of dimensions m and n respectively. Let $f\colon M \to N$ and $g\colon N \to M$ be C^∞ functions. If $f \circ g$ is the identity, then what can you conclude about the possible values of m and n? What about if $g \circ f$ is the identity?

2. Exercise II.6.7

3. Exercise III.1.3

4. Let $f(t) = (t(1-t), t^2(1-t))$. Find all real numbers a and b satisfying $a < b$ such that $f\colon (a,b) \to \mathbb{R}^2$ is an injective immersion but not an embedding.

5. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ function, and define $g\colon \mathbb{R}^n \to \mathbb{R}^{n+m}$ by $g(x) = (x, f(x))$. Show that g is an embedding.

6. Let $M_{mn}(\mathbb{R})$ be the space of all real $m \times n$ matrices and let $M^k_{mn}(\mathbb{R})$ be the subset of all those $m \times n$ matrices with rank k. Show that $M^k_{mn}(\mathbb{R}) \subset M_{mn}(\mathbb{R})$ is a regular submanifold. What is its dimension?

   Hint: Use the hint to exercise 13, section 4, page 27 of Guillemin and Pollack's *Differential Topology*: `https://www.cimat.mx/~gil/docencia/2020/topologia_diferencial/5BGuillemin,Pollack%5DDifferential_Topology(1974).pdf` to check the definition from Boothby.

7. Let $a > 0$, and let C be the set of points in the xy coordinate plane of $\mathbb{R}^3$ which are distance a from the origin. Let $b \in (0,a)$, and let M be the set of points in $\mathbb{R}^3$ which are distance b from C. Prove that M is a regular submanifold of $\mathbb{R}^3$.

8. Let c and d be given positive numbers, and let $N = \{(w,x,y,z) \in \mathbb{R}^4 \colon w^2 + x^2 = c^2 \text{ and } y^2 + z^2 = d^2\}$. Prove that N is a regular submanifold of $\mathbb{R}^4$, and find a diffeomorphism between N and the manifold M from the previous problem.

9. Let $\theta\colon \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ be the one-parameter group acting on $\mathbb{R}^2$ given by

$$(t,x,y) \mapsto \theta_t(x,y) = (x\cos t + y \sin t, y\cos t - x\sin t).$$

   Let X_p be the infinitesimal generator of θ at $p = (1,3)$, and let $f(x,y) = 5x - 7y$. Find $X_p f$.