MA 562 first midterm review problems
Version as of October 28th.

The second midterm will be in class on Monday October 31st. Most of the exam will be closely based on problems, or on parts of problems, from the list below. Please let me know if you have a question or find a mistake.

1. Exercise IV.4.3. *Hint:* What is $\exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)$?

2. Do the above problem with $X = x^k(\partial/\partial x) + xy(\partial/\partial y)$ where $k \in \{0, 1, 2, \dots\}$ is given.

3. Exercise V.4.2. Show by way of example that the conclusion is invalid if the word 'closed' is deleted.

4. Exercise V.4.5.

5. Let $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (u(x, y), v(x, y)) = (x^2 + y, xy)$. Evaluate Df, $f_*(\partial_x)$, $f_*(\partial y)$, $f^*(du)$, and $f^*(dv)$, each at a given point $(x, y) = (a, b)$.

6. Problems 4–7 from Chapter 1 of do Carmo's book: `https://link-springer-com.ezproxy.lib.purdue.edu/book/10.1007/978-3-642-57951-6` . For number 5, only compute the wedges and not the d's.

7. Let $M \subset \mathbb{R}^n$ be the sphere of radius R centered at 0, equipped with the induced Riemannian metric.

   (a) Find a smooth vector field $V = \sum_{j=1}^n V^j \partial_{x^j}$ on $\mathbb{R}^n$ which is unit normal to M, i.e. $\sum_{j=1}^n (V^j(x))^2 = 1$ for every $x \in M$ and $\sum_{j=1}^n V^j(x)W^j(x) = 0$ for every $x \in M$ and $\sum_{j=1}^n W^j(x)\partial_{x^j} \in T_xM$.

   (b) Let $\alpha = i(V)\Omega$, where $(i(V)\Omega)(V_1, \dots, V_{n-1}) = \Omega(V, V_1, \dots, V_{n-1})$ and $\Omega = dx^1 \wedge \cdots \wedge dx^n$. Find $a_1(x), \dots, a_n(x)$ such that $\alpha = a_1(x)dx^2 \wedge \cdots \wedge dx^n + \cdots + a_n(x)dx^1 \wedge \cdots \wedge dx^{n-1}$.

   (c) Show that α is the volume form on M with respect to some orientation on M.