Kiril Datchev
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Quiz 7

Name:_____

Under certain hypotheses, given $f \in L^2(\mathbb{R})$, the equation

$$-h^2 u'' + Vu - Eu = f$$

has a unique solution $u \in L^2(\mathbb{R})$, given by

$$u(x) = \int_{-\infty}^{\infty} K(x,y) f(y)\, dy, \qquad \text{where} \qquad K(x,y) = -\frac{1}{h^2 W} \psi_L(\min(x,y)) \psi_R(\max(x,y)),$$

and ψ_L and ψ_R solve

$$-h^2 \psi'' + V\psi - E\psi = 0, \qquad \text{with} \qquad \lim_{x \to -\infty} \psi_L(x) = \lim_{x \to +\infty} \psi_R(x) = 0.$$

1. Give a formula for W, and list some of the hypotheses.
2. How does K simplify in the case $h = 1$, $V = 0$, $E = -1$?

Solutions:

1. $W = \psi_L' \psi_R - \psi_L \psi_R'$. A simple set of sufficient technical hypotheses is $V \in C^2(\mathbb{R})$, $E < \inf V$, $\int_{-\infty}^{\infty} a|a''| < \infty$, where $a = (V - E)^{-1/4}$.
2. We can use $\psi_L(x) = e^x$, $\psi_R(x) = e^{-x}$, $W = 2$, and get

$$K(x, y) = -\frac{1}{2} e^{-\min(x,y)+\max(x,y)} = -\frac{1}{2} e^{-|x-y|}$$