MA 598: Wave Equations
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In this course we will study wave equations, beginning with the simplest (D'Alembert's original) $u_{tt} = c^2 u_{xx}$ where the wavespeed c is a constant. We will then consider the Schrödinger equation $iu_t = -u_{xx}$ and the higher dimensional generalizations of both. We will derive solution formulas, travel speeds, the Huygens principle, propagation of singularities, and wave decay rates.

We will then use the methods of microlocal and semiclassical analysis, i.e. particle–wave duality, to examine analogs and generalizations of these results for other wave and Schrödinger equations, including allowing the wavespeed c to vary depending on the position and direction of the wave, and adding a potential energy term $V(x)u$ on the right hand side of the equation.

Spoiler alert: the general conclusion is the following. Behavior of solutions to the wave equation with wavespeed $c(x)$ is governed by the geometry of the trajectories of particles traveling with speed $c(x)$ according to Fermat's principle of least time. Behavior of solutions to the Schrödinger equation with potential energy $V(x)$ is governed by the geometry of the trajectories of particles traveling according to Newton's law $F = ma$ with force $F(x) = -V'(x)$. Both kinds of trajectories are described in a unified way by Hamilton's action principle, and the theories of microlocal and semiclassical analysis make the connection with wave evolution. More specifically, salient features of waves, especially singularities, follow these trajectories as they evolve.

As mentioned above, the simplest examples are waves which solve $u_{tt} = c^2 u_{xx}$. These waves take the form $u(x,t) = f(x + ct) + g(x - ct)$, where the f term corresponds to a particle traveling to the left at speed c, and the g term corresponds to a particle traveling to the right at speed c. In other examples we can usually describe neither the waves nor the particle trajectories so simply, but we instead relate more accessible major aspects of wave and particle behavior to one another. Especially important for waves are bound states, which live forever, and resonances, which can live for a long time. These correspond to particle trajectories which stay always in some bounded region.

For our work we will develop tools from distribution theory, Fourier analysis, Sobolev spaces, pseudodifferential operators, and scattering theory. These tools are also useful for the study of more general differential equations, and we will touch on such connections as we go. The course is intended to be accessible to students coming from a range of backgrounds and will assume knowledge only of real and complex analysis at the level of 440/504 and 425/525.

Sources for the material include Friedlander and Joshi's *Introduction to the Theory of Distributions*, Taylor's *Partial Differential Equations*, Zworski's *Semiclassical Analysis*, and Dyatlov and Zworski's *Mathematical Theory of Scattering Resonances*, but our treatment will be less advanced: notes will be available at the course website `https://www.math.purdue.edu/~kdatchev/598/598.htm`.