

Free waves

1. Free waves in one dimension. We begin by deriving D'Alembert's wave equation for a vibrating string.

Let ρ be the mass density of the string and τ be the tension. Let $u(x, t)$ be the vertical displacement of the string at position x and time t , and let $\alpha(x, t) = \arctan(u_x(x, t))$ be the angle of displacement at position x and time t . Let I be the segment of string from x to $x + h$.

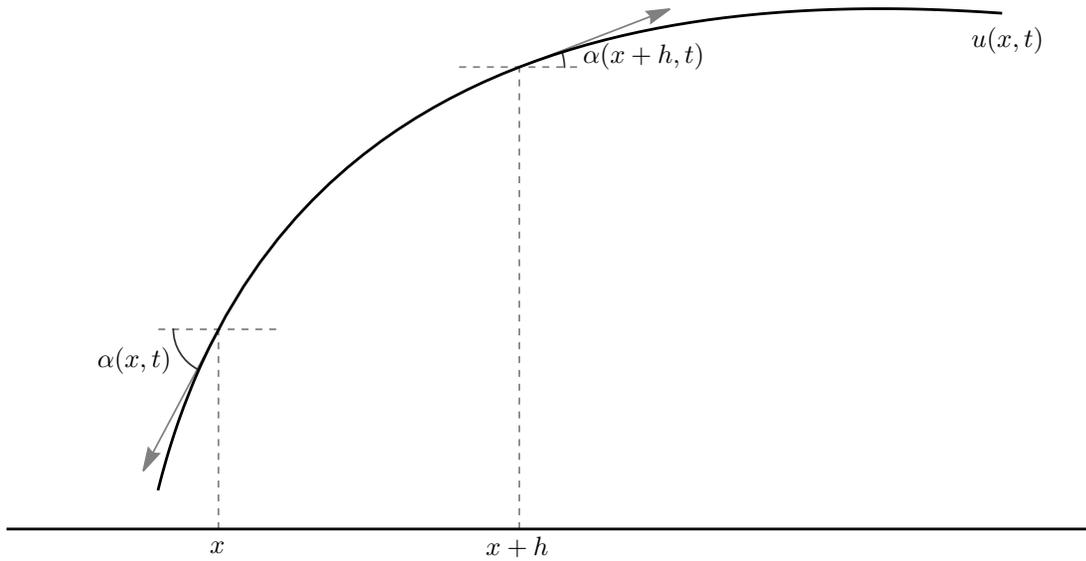


FIGURE 1. A portion of the vibrating string

Setting the derivative of the vertical momentum of I equal to the vertical force on I gives

$$\int_I \rho u_{tt}(y, t) ds = \tau \sin \alpha(x + h, t) - \tau \sin \alpha(x, t), \quad (1)$$

where the integral on the left is a line integral. We write this line integral out as

$$\int_I \rho u_{tt}(y, t) ds = \int_x^{x+h} \rho u_{tt}(y, t) \sqrt{1 + u_x^2(y, t)} dy. \quad (2)$$

Plugging (2) into (1), dividing by h and letting $h \rightarrow 0$ gives

$$\rho u_{tt}(x, t) \sqrt{1 + u_x^2(x, t)} = \tau \partial_x \sin(\alpha(x, t)) = \tau \partial_x \frac{u_x(x, t)}{\sqrt{1 + u_x^2(x, t)}}$$

Using the binomial series $(1 + u_x^2)^s = 1 + s u_x^2 + \frac{s(s-1)}{2} u_x^4 + \dots$, gives

$$\rho u_{tt} (1 + \frac{1}{2} u_x^2 + \dots) = \tau \partial_x (u_x (1 - \frac{1}{2} u_x^2 + \dots)).$$

Kiril Datchev, May 20, 2024. These notes are under development. Questions, comments, and corrections are gratefully received at kdatchev@purdue.edu.

If u_x is small enough to be negligible, i.e. if the vibrations are small, we get

$$\rho u_{tt} = \tau u_{xx}.$$

Divide through by ρ , put $c = \sqrt{\tau/\rho}$, to get

$$\partial_t^2 u - c^2 \partial_x^2 u = 0. \quad (3)$$

By the chain rule we see that this is solved by

$$u(x, t) = f(x + ct), \quad \text{and} \quad u(x, t) = g(x - ct),$$

whenever f and g are twice differentiable, and any superposition

$$u(x, t) = f(x + ct) + g(x - ct), \quad (4)$$

is also a solution. We have a component, or signal, traveling to the left at speed c and one traveling to the right at speed c .

EXERCISE 1. Suppose $u(x, t)$ obeys (4) and that the initial position $u(x, 0) = \varphi(x)$ and velocity $u_t(x, 0) = \psi(x)$ are given. Derive a formula for u in terms of φ and ψ . This formula is called *d'Alembert's formula*. Note that u is uniquely defined even though f and g are not. What regularity assumption do you need on φ and ψ ?

To see that *any* solution of (3) has the form (4), we use a *change of variable* or *change of coordinates*.

THEOREM 1. *Let $u = u(x, t)$ be twice differentiable and let c be a positive real number. Then u solves (3) if and only if there are twice differentiable functions f and g such that (4) holds.*

Proof. As mentioned previously, (4) implies (3) by the chain rule. To prove the converse, put $r = x + ct$ and $s = x - ct$. Then $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial s}{\partial x} \frac{\partial}{\partial s}$, and similarly for $\frac{\partial}{\partial t}$, giving

$$\partial_x = \partial_r + \partial_s, \quad \partial_t = c\partial_r - c\partial_s,$$

or

$$0 = \partial_t^2 u - c^2 \partial_x^2 u = c^2(\partial_r^2 + \partial_s^2 - 2\partial_r \partial_s)u - c^2(\partial_r^2 + \partial_s^2 + 2\partial_r \partial_s)u = -4c^2 \partial_r \partial_s u.$$

Dividing out the $-4c^2$ and integrating with respect to r shows that

$$\partial_s u(x(r, s), t(r, s)) = G(s),$$

for some function G which is once differentiable. Integrating next with respect to s shows that

$$u(x(r, s), t(r, s)) = f(r) + \int G(s),$$

for some function f which is twice differentiable, and this reduces to (4). \square

To see the particle–wave correspondence, observe that by Newton's second law, the derivative of the momentum of a classical particle is equal to the force on it. Thus, a free particle moves with a constant momentum, i.e. a constant velocity. Its trajectory is given by $q: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$q''(t) = 0, \quad q(t) = at + b, \quad (5)$$

where $a \in \mathbb{R}$ is its velocity and $b \in \mathbb{R}$ is its initial position. For particles of light, i.e. photons, the same equation holds for the trajectory except that $|a| = c = 3 \cdot 10^8$ meters per second.

The same equation (3) holds for light waves traveling through xyz space, provided they are constant in the y and z directions. This is a good approximation for light waves from the sun reaching the earth. Then equation (4) shows that each such wave (called a plane wave) is a superposition of two components, each following a corresponding particle trajectory (5).

We have thus found our first instance of particle–wave correspondence: the relationship between wave solutions $u(x, t)$ and particle solutions $q(t)$ is universal, i.e. independent of the parameter b and the functions f and g .

2. Distribution theory. To have a solution to the wave equation (3) in the traditional sense, as in our derivation above, u must be twice differentiable. This restriction on u causes trouble, both in the development of the theory and in applications. We want to allow

$$u(t, x) = \begin{cases} 1, & |t - x| \leq 1, \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

and other similar examples, as well as more singular ones, such as waves localized at a single point. Distribution theory provides a general framework for making sense of (6) as a solution to the wave equation (3).

We will understand such singular solutions as limits of smooth ones.

EXAMPLE 1. For example, to make sense of the derivative of $\operatorname{sgn} x$, observe that

$$u_n(x) = \operatorname{erf}(nx) = \frac{2}{\sqrt{\pi}} \int_0^{nx} e^{-t^2} dt \rightarrow \operatorname{sgn} x, \quad \text{as } n \rightarrow \infty.$$

The derivative of $\operatorname{sgn} x$ will be given by the limit of the sequence $u'_n(x)$. We will call this limit $2\delta_0(x)$, i.e. 2 times the Dirac delta at 0, because it corresponds to a jump of 2 located at $x = 0$. Similarly, the second derivative of $\operatorname{sgn} x$ will be given by the limit of the sequence $u''_n(x)$, and is called $2\delta'_0(x)$. The Dirac delta δ_0 is also called the unit mass at the origin, and its derivative δ'_0 is called the unit dipole at the origin. The word *unit* refers to the fact that $\frac{1}{2} \int_{-\infty}^{\infty} u'_n(x) dx = 1$ for every n , so we can say that

$$\int_{-\infty}^{\infty} \delta_0 = 1. \quad (7)$$

We sometimes write more shortly δ_0 as δ .

To give a precise meaning to the sense in which these limits exist, and to objects such as δ_0 and δ'_0 and equations such as (7), we use the concept of *duality* from linear algebra. Recall that if V is a finite dimensional vector space over \mathbb{C} , then its *dual* is the set of linear functionals $V \rightarrow \mathbb{C}$. We may identify V with its dual using an inner product: for example the vector $(3, 6i, 7)$ can be identified with the functional

$$(x, y, z) \mapsto 3x + 6iy + 7z,$$

where (x, y, z) ranges over \mathbb{C}^3 . Similarly, we will identify a function g with a functional

$$\varphi \mapsto g[\varphi] = \int g\varphi,$$

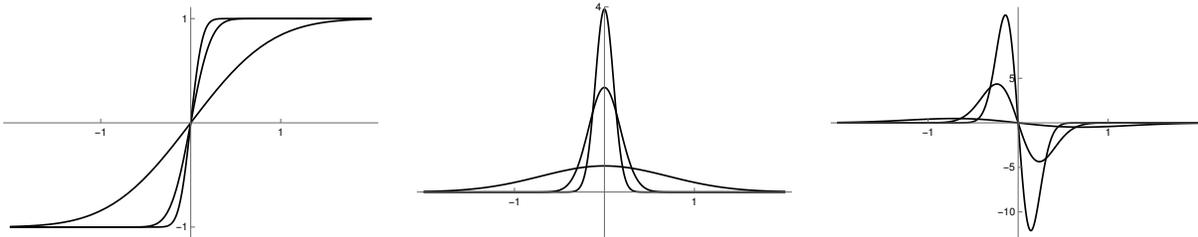


FIGURE 2. Graphs of $u_n(x) = \operatorname{erf}(nx)$ and its first two derivatives, for $n = 1, 3,$ and 5 . As $n \rightarrow \infty$, $u_n \rightarrow \operatorname{sgn}$, $u'_n \rightarrow 2\delta_0$, and $u''_n \rightarrow 2\delta'_0$.

with φ ranging over a suitable space of functions, called test functions. A space of *distributions*, or *generalized functions*, will be a more general space of such linear functionals. This allows one to define and analyze operations on distributions (which may be ‘nasty’ objects such as δ_0 or δ'_0) in terms of operations on test functions (which are ‘nice’ objects).

Thus, in Example 1, we will have

$$u_n[\varphi] = \int \operatorname{erf}(nx)\varphi(x)dx \rightarrow \int \operatorname{sgn}(x)\varphi(x)dx, \quad (8)$$

provided φ is a nice enough function that the limit can pass under the integral. Similarly, by integration by parts,

$$u'_n[\varphi] = \int u'_n(x)\varphi(x)dx = - \int u_n(x)\varphi'(x)dx \rightarrow - \int \operatorname{sgn}(x)\varphi'(x)dx = 2\varphi(0), \quad (9)$$

provided φ is nice enough that the boundary terms at $\pm\infty$ vanish. Similarly again,

$$u''_n[\varphi] = \int u''_n(x)\varphi(x)dx = - \int u'_n(x)\varphi'(x)dx = -u'_n[\varphi'] \rightarrow -2\varphi'(0). \quad (10)$$

Thus we define

$$\delta_0[\varphi] = \varphi(0), \quad \delta'_0[\varphi] = -\varphi'(0). \quad (11)$$

Various spaces of test functions φ are ‘nice enough’ that the above manipulations are justified. Because Fourier analysis will be central to all our work, the best space of test functions for us will be the space of Laurent Schwartz [Sch] (as we will see later, this space is preserved by the Fourier transform).

DEFINITION 1. We denote the space of infinitely differentiable functions $\mathbb{R}^d \rightarrow \mathbb{C}$ by $C^\infty(\mathbb{R}^d)$. A function in $C^\infty(\mathbb{R}^d)$ is a *Schwartz function* if it and all its derivatives are $O(|x|^{-N})$ for every N , as $|x| \rightarrow \infty$. The space of Schwartz functions is denoted $\mathcal{S}(\mathbb{R}^d)$ or \mathcal{S} for short.

If $\varphi \in \mathcal{S}$ then the convergence in (8)–(10) follows from the dominated convergence theorem.¹

The fundamental example of an element of $\mathcal{S}(\mathbb{R})$ is the Gaussian $x \mapsto e^{-x^2}$. Note that if $\varphi(x) \in \mathcal{S}(\mathbb{R})$, then so are $\varphi'(x)$ and $x\varphi(x)$, as well as $\varphi(ax + b)$ for any $a \in \mathbb{R} \setminus \{0\}$ and $b \in \mathbb{R}$.

¹The *dominated convergence theorem* says that if $f_n(x) \rightarrow f(x)$ and there is an integrable g such that $|f_n(x)| \leq g(x)$ for all n , then $\int f_n \rightarrow \int f$; see e.g Theorem 2.24 of [Fol]. In this case we use $f_n(x) = \operatorname{erf}(nx)\varphi(x)$, $f(x) = \operatorname{sgn}(x)\varphi(x)$, $g(x) = |\varphi(x)|$.

Moreover \mathcal{S} is closed under \mathbb{C} -linear combinations and under products. Of course \mathcal{S} has many more elements but the ones described above are already enough to go on with.

Thus in equations (8)–(11) the expressions $T[\varphi]$ mean that T is a linear functional $\mathcal{S} \rightarrow \mathbb{C}$. More generally, we make the following

DEFINITION 2. To any function $f \in \mathcal{S}(\mathbb{R}^d)$ we associate a mapping $\mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ defined by

$$f[\varphi] = \int f\varphi.$$

We say that a sequence of mappings $u_n: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ converges in the sense of distributions to $u: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ if

$$u_n[\varphi] \rightarrow u[\varphi], \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d).$$

A \mathbb{C} -linear mapping $u: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$ is a *tempered distribution*, and write $u \in \mathcal{S}'(\mathbb{R}^d)$ or $u \in \mathcal{S}'$ for short, if it is the limit in the sense of distributions of a sequence of functions in $\mathcal{S}(\mathbb{R}^d)$.

Thus δ_0 and δ'_0 are tempered distributions. For any specific tempered distribution T , the mapping $\varphi \mapsto T[\varphi]$ generally extends naturally to a much larger space than \mathcal{S} . For example, in the case of δ_0 , we may take φ to be the constant function 1, and then (7) just means $\delta_0[1] = 1$. One also often writes the equation $\delta[\varphi] = \varphi(0)$ using more familiar notation as

$$\int_{-\infty}^{\infty} \delta(x)\varphi(x)dx = \varphi(0). \quad (12)$$

The point is that (12) (which is known as the *sifting property*) is the *defining property* of δ , with the understanding that the integral in (12) is not a Riemann integral or a Lebesgue integral but a new kind of integral, a *distributional* integral.

We will generally omit the adjective ‘tempered’ below since all of our distributions will be tempered. See the Further Discussion below for some other kinds of distributions.

Note that the sequence $\text{erf}(nx)$ itself does not fit our definition, because the terms are not Schwartz functions. This can be remedied by multiplying each term by another sequence of Schwartz functions, tending to the constant function 1. For example, define

$$u_{n,m} = \text{erf}(nx)e^{-x^2/m}.$$

Then, for each n , $u_{n,m} \rightarrow \text{erf}(nx)$ as $m \rightarrow \infty$, and $u_{n,n}$ is a sequence in \mathcal{S} which converges to $\text{sgn } x$.

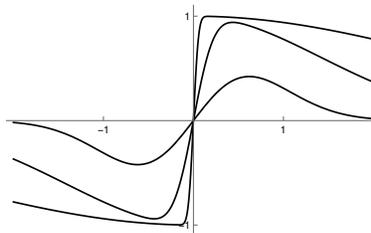


FIGURE 3. Graph of $u_{n,n}(x) = \text{erf}(nx)e^{-x^2/n}$ for $n = 1, 4$, and 16 .

In the same way, the Heaviside function $H(x) = \frac{1}{2}(1 + \operatorname{sgn} x)$ is the limit of the sequence $\frac{1}{2}(1 + \operatorname{erf}(nx))e^{-x^2/n}$.

EXERCISE 2. Find a sequence of Schwartz functions converging in the sense of distributions to each of the following functions:

- (1) The function $|x|$.
- (2) The characteristic function $\mathbf{1}_{[a,b]}$ of a given interval $[a, b]$, i.e. the function which is one inside the interval and zero outside it.

Hint: Write the functions in terms of the Heaviside and/or sgn function.

Just as δ_0 was defined as the limit of the sequence $ne^{-n^2x^2}/\sqrt{\pi}$ in the sense of distributions, for $a \in \mathbb{R}$ we define δ_a as the limit of the sequence $ne^{-n^2(x-a)^2}/\sqrt{\pi}$. We could just as easily use another sequence of integrable functions which integrate to 1 as a basis for the definition, as in the exercise below. This is analogous to how e is the limit of both $(1 + \frac{1}{n})^n$ and $\sum_{j=0}^n 1/j!$.

EXERCISE 3. Let $u_n(x) = an^b/(1 + n^2x^2)$, where a and b are real numbers. For which values of a and b does u_n converge in the sense of distributions to δ_0 ? Bonus question: what happens for other values of a and b ?

We define Dirac deltas in higher dimensions in the same way. Given $a \in \mathbb{R}^d$, let δ_a be the limit of the sequence $n^d e^{-\pi n^2|x-a|^2}$. This is the unit mass situated at $x = a$.

The calculations in equations (8)–(11) also motivate the definition of derivative of a distribution.

DEFINITION 3. Given a distribution $u: \mathcal{S}(\mathbb{R}) \rightarrow \mathbb{C}$, we define its *distributional derivative* u' by

$$u'[\varphi] = -u[\varphi'].$$

We similarly define $\partial_{x_j} u[\varphi] = -u[\partial_{x_j} \varphi]$ for a distribution $u: \mathcal{S}(\mathbb{R}^d) \rightarrow \mathbb{C}$.

It is significant that *every* distribution has a distributional derivative. Moreover, by definition, if $u_n \rightarrow u$ in the sense of distributions, then $u'_n \rightarrow u'$.

EXAMPLE 2. If $u = 2\delta_3 - 4\delta_5$, then $u[\varphi] = 2\varphi(3) - 4\varphi(5)$, and $u' = 2\delta'_3 - 4\delta'_5$, $u'[\varphi] = -2\varphi'(3) + 4\varphi'(5)$. Also $\delta(x)$ is the derivative of the Heaviside function $H(x)$, which is the derivative of $xH(x)$.

EXERCISE 4. Find the first two distributional derivatives of the functions $|x|$ and $\mathbf{1}_{[a,b]}$ from Exercise 2 above.

We say a distribution u is a distributional solution to the wave equation (3) if

$$u[\varphi_{tt} - c^2\varphi_{xx}] = 0,$$

for every $\varphi \in \mathcal{S}(\mathbb{R}^2)$. If u_n is a sequence of distributional solutions converging to u in the sense of distributions, then u is a distributional solution as well. For instance, the sequence

$u_n = ne^{-n^2(x-ct)^2}/\sqrt{\pi}$ converges to a distributional solution denoted $u = \delta(x - ct)$. To compute it, use the change of variables $r = x + ct$, $s = x - ct$ and then $y = ns$ to write

$$\begin{aligned} u_n[\varphi] &= \frac{n}{\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-n^2(x-ct)^2} \varphi(x, t) dx dt = \frac{n}{2c\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-n^2 s^2} \varphi\left(\frac{r+s}{2}, \frac{r-s}{2c}\right) ds dr \\ &= \frac{1}{2c\sqrt{\pi}} \int_{\mathbb{R}^2} e^{-y^2} \varphi\left(\frac{r+(y/n)}{2}, \frac{r-(y/n)}{2c}\right) dy dr \rightarrow \frac{1}{2c} \int_{\mathbb{R}} \varphi\left(\frac{r}{2}, \frac{r}{2c}\right) dr = u[\varphi]. \end{aligned} \quad (13)$$

Thus we may write $\delta(x - ct) = \frac{1}{2c}\delta(s)$ as a distribution on \mathbb{R}^2 , with respect to the (r, s) variables; note that the factor of $1/2c$ comes from the Jacobian in the change of variables formula.

EXERCISE 5. Redo the calculation in (13), but instead of the variables r and s , use the variables $\tilde{r} = x + at$, $s = x - ct$, where a is a real constant. Find a constant b such that $\delta(x - ct) = b\delta(s)$ as a distribution on \mathbb{R}^2 , with respect to the (\tilde{r}, s) variables. Denote by L the line $x = ct$, and find a constant m such that $\delta(x - ct) = m\delta_L$, where $\delta_L[\varphi] = \int_L \varphi$ and \int_L is the line integral over L .

EXERCISE 6. Find a sequence of C^∞ solutions to (3) which converge in the sense of distributions to u , where u is given by (6). Deduce that u is a distributional solution of the wave equation (3) with $c = 1$, and so are u_x and u_t . Find the corresponding sequence of C^∞ functions converging to u_x in the sense of distributions, and plot part of the sequence.

3. The Fourier transform. For our further study of waves, our central tool will be the Fourier transform. Recall that the free wave and Schrödinger equations in dimension d are given by

$$u_{tt} - \Delta u = 0, \quad iu_t + \Delta u = 0,$$

where $\Delta u = \nabla^2 u = \sum_{j=1}^d u_{x_j x_j}$. We wish to write solutions to these as superpositions of sinusoidal waves, namely

$$u(x, t) = \int_{\mathbb{R}^d} a(\xi) e^{i\xi x} e^{-i|\xi|t} d\xi + \int_{\mathbb{R}^d} b(\xi) e^{i\xi x} e^{i|\xi|t} d\xi, \quad u(x, t) = \int_{\mathbb{R}^d} a(\xi) e^{i\xi x} e^{-i\xi t} d\xi.$$

The basic problem, solved by Fourier [Fou], is to write a given function φ as a superposition

$$\varphi(x) = \int_{\mathbb{R}^d} e^{ix\xi} a(\xi) d\xi,$$

for some suitable function a .

To get a feel for the problem we start with

$$f(x) = \int_{-\infty}^{\infty} e^{ix\xi} e^{-\xi^2} d\xi, \quad (14)$$

and use Feynman's trick of differentiating under the integral sign [Fey] followed by integration by parts to get

$$f'(x) = \int i\xi e^{ix\xi} e^{-\xi^2} d\xi = \frac{-i}{2} \int e^{ix\xi} \frac{d}{d\xi} e^{-\xi^2} d\xi = \frac{i}{2} \int \partial_\xi (e^{ix\xi}) e^{-\xi^2} d\xi = -\frac{x}{2} f(x).$$

Interchanging the derivative and integral in the first equality is justified because the derivative with respect to x of the integrand is bounded in absolute value by $|\xi|e^{-\xi^2}$, and $\int |\xi|e^{-\xi^2} < +\infty$.²

²This follows from the dominated convergence theorem; see e.g. [Fol, Theorem 2.27].

Solving $f'(x) = -xf(x)/2$ by separation of variables gives $f(x) = f(0)e^{-x^2/4}$, and from (14) we get $f(0) = \sqrt{\pi}$, giving

$$\sqrt{\pi}e^{-x^2/4} = \int e^{ix\xi}e^{-\xi^2}d\xi,$$

or, changing variables,

$$\frac{n}{\sqrt{\pi}}e^{-n^2(x-y)^2} = \frac{n}{\pi} \int e^{2in(x-y)\xi}e^{-\xi^2}d\xi = \frac{1}{2\pi} \int e^{ix\xi}e^{-iy\xi}e^{-\xi^2/4n^2}d\xi. \quad (15)$$

The reason for the form of the left hand side of (15) is that

$$\frac{n}{\sqrt{\pi}}e^{-n^2(x-y)^2} \rightarrow \delta(x-y) \quad (16)$$

in the sense of distributions. Now let $\varphi \in \mathcal{S}$, and write

$$\begin{aligned} \varphi(x) &= \lim_{n \rightarrow \infty} \frac{n}{\sqrt{\pi}} \int \varphi(y)e^{-n^2(x-y)^2}dy = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int \varphi(y) \int e^{ix\xi}e^{-iy\xi}e^{-\xi^2/4n^2}d\xi dy \\ &= \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int e^{-\xi^2/4n^2}e^{ix\xi} \int e^{-iy\xi}\varphi(y)dy d\xi = \lim_{n \rightarrow \infty} \frac{1}{2\pi} \int e^{-\xi^2/4n^2}e^{ix\xi}\hat{\varphi}(\xi)d\xi = \frac{1}{2\pi} \int e^{ix\xi}\hat{\varphi}(\xi)d\xi. \end{aligned} \quad (17)$$

Here in the first equality we used (16), in the second equality we used (15), in the third equality the switch of the order of integration is justified by Fubini's theorem³ because the double integral is absolutely convergent $\int \int |e^{-\xi^2/4n^2}\varphi(y)|dyd\xi = \int e^{-\xi^2/4n^2}d\xi \int |\varphi(y)|dy < +\infty$, in the fourth equality we used the definition

$$\hat{\varphi}(\xi) = \int e^{-iy\xi}\varphi(y)dy, \quad (18)$$

and in the fifth equality we used the fact that if $\varphi \in \mathcal{S}$, then $\hat{\varphi} \in \mathcal{S}$. Let us elaborate on this last fact:

THEOREM 2. *If $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$. Moreover*

$$\xi_j \hat{\varphi}(\xi) = -i\widehat{\partial_{x_j}\varphi}(\xi), \quad \partial_{\xi_j}\hat{\varphi}(\xi) = -i\widehat{(x_j\varphi)}(\xi) \quad (19)$$

Proof. We begin with the case $d = 1$. Then integrating by parts shows that

$$\xi \hat{\varphi}(\xi) = \int i\partial_y(e^{-iy\xi})\varphi(y)dy = ie^{-iy\xi}\varphi(y)\Big|_{y=-\infty}^{\infty} - i \int e^{-iy\xi}\varphi'(y)dy = 0 - i\hat{\varphi}(\xi),$$

and differentiating under the integral sign shows that

$$\widehat{(x\varphi)}(\xi) = \int i\partial_\xi(e^{-iy\xi})\varphi(y)dy = i\partial_\xi \int e^{-iy\xi}\varphi(y)dy.$$

That proves (19) in the case $d = 1$. If $d > 1$, write $\int e^{-iy\xi}\varphi(y)dy$ as an iterated integral with dy_j innermost, and do the same calculation on the one-dimensional integral with respect to y_j .

³*Fubini's Theorem* says that if a double integral is absolutely convergent, then its value is equal to that of any corresponding iterated integral; see e.g. Theorem 2.37 of [Fol].

To prove $\hat{\varphi} \in \mathcal{S}(\mathbb{R}^d)$, again begin with the case $d = 1$. For any nonnegative integers k and N we have

$$\begin{aligned}\xi^N \hat{\varphi}^{(k)}(\xi) &= \int \xi^N e^{-iy\xi} (iy)^k \varphi(y) dy = \int (\partial_y^N e^{-iy\xi}) i^N (iy)^k \varphi(y) dy \\ &= \int e^{-iy\xi} (-i)^{N+k} \partial_y^N (y^k \varphi(y)) dy.\end{aligned}$$

Thus

$$|\xi^N \hat{\varphi}^{(k)}(\xi)| \leq \int |\partial_y^N (y^k \varphi(y))| dy.$$

If $d > 1$, use the same proof, replacing ξ^N by an arbitrary monomial and $\hat{\varphi}^{(k)}$ with an arbitrary partial derivative of $\hat{\varphi}$. \square

The function $\hat{\varphi}$ defined by (18) is called the *Fourier transform* of φ .⁴ Thus, the calculation (17) has shown the Fourier inversion theorem for Schwartz functions:

THEOREM 3. *If $\varphi \in \mathcal{S}(\mathbb{R}^d)$, then*

$$\varphi(x) = \frac{1}{(2\pi)^d} \int e^{ix\xi} \hat{\varphi}(\xi) d\xi. \quad (20)$$

In other words, $\hat{\hat{\varphi}}(x) = (2\pi)^d \varphi(-x)$.

Proof. If $d = 1$ then this is the calculation (17). If $d > 1$, use the same calculation one variable at a time. For example, if $d = 2$, then

$$\begin{aligned}\varphi(x_1, x_2) &= \frac{1}{2\pi} \int e^{ix_1 \xi_1} \int e^{-iy_1 \xi_1} \varphi(y_1, x_2) dy_1 d\xi_1 \\ &= \frac{1}{(2\pi)^2} \int e^{ix_1 \xi_1} \int e^{ix_2 \xi_2} \int e^{-iy_1 \xi_1} \varphi(y_1, y_2) dy_1 dy_2 d\xi_2 d\xi_1 = \frac{1}{(2\pi)^d} \int e^{ix\xi} \hat{\varphi}(\xi) d\xi,\end{aligned}$$

where in the first equality we applied the case $d = 1$ to $x_1 \mapsto \varphi(x_1, x_2)$ and in the second we applied it to $x_2 \mapsto \int e^{-iy_1 \xi_1} \varphi(y_1, x_2) dy_1$. \square

To define the Fourier transform of a tempered distribution, we use a formula analogous to the one we used to define the distributional derivative. More specifically, just as the integration by parts formula $\int u' \varphi = -\int u \varphi'$ led us to define $u'[\varphi] = -u[\varphi']$, so the formula

$$\int u(x) \hat{\varphi}(x) dx = \int u(x) \int e^{-ixy} \varphi(y) dy dx = \int \varphi(y) \int e^{-ixy} u(x) dx dy = \int \hat{u}(y) \varphi(y) dy \quad (21)$$

leads us to define

$$\hat{u}[\varphi] = u[\hat{\varphi}]. \quad (22)$$

As a consequence, if $u_n \rightarrow u$ in the sense of distributions, then $\hat{u}_n \rightarrow \hat{u}$ in the sense of distributions, and moreover

$$\xi_j \hat{u} = \widehat{-i \partial_{x_j} u}, \quad \partial_{\xi_j} \hat{u} = \widehat{-ix_j u}.$$

⁴Our convention here, that the Fourier transform of φ is given by $\int e^{-ix\xi} \varphi(x) dx$, is common and convenient in the theory of differential equations. The two other most common conventions are $\frac{1}{\sqrt{2\pi}} \int e^{-ix\xi} \varphi(x) dx$ [Tit] and $\int e^{-2\pi ix\xi} \varphi(x) dx$ [Sch].

We also have the formula

$$\hat{u}(x) = (2\pi)^d u(-x). \quad (23)$$

EXERCISE 7. Show that if u is a distribution given by $u[\varphi] = \int f\varphi$ for some integrable function (i.e. $\int |f| < +\infty$) then \hat{u} is a distribution given by $\hat{u}[\varphi] = \int g\varphi$, where $g(\xi) = \int e^{-iy\xi} f(y) dy$. This proves that we can calculate the Fourier transform of a function in the sense of distributions using the formula (18) as long as the function is integrable (it doesn't have to be Schwartz).

EXAMPLE 3.

(1) By the calculation in (15), we have

$$\widehat{e^{-b|x|^2}}(\xi) = \left(\frac{\pi}{b}\right)^{d/2} e^{-|\xi|^2/4b}.$$

(2) Applying the definitions (22) and (18), we have

$$\delta[\varphi] = \delta[\hat{\varphi}] = \hat{\varphi}(0) = \int e^{-i0y} \varphi(y) dy = 1[\varphi].$$

and so $\hat{\delta} = 1$. We can also get this using sequences of Gaussians: we have $\frac{n^d}{\sqrt{\pi^d}} e^{-n^2|x|^2} \rightarrow \delta$ and

$$\frac{n^d}{\sqrt{\pi^d}} \widehat{e^{-n^2|x|^2}}(\xi) = e^{-|\xi|^2/4n^2} \rightarrow 1.$$

(3) From $\hat{\delta} = 1$ and (23), we get

$$\hat{1}(\xi) = (2\pi)^d \delta(-\xi) = (2\pi)^d \delta(\xi).$$

Sometimes this formula is also written

$$\delta(\xi) = \frac{1}{(2\pi)^d} \int e^{-ix\xi} dx,$$

and we can understand this integral as meaning

$$\delta(\xi) = \lim_{n \rightarrow \infty} \frac{1}{(2\pi)^d} \int e^{-ix\xi} e^{-x^2/n} dx, \quad (24)$$

with the limit taken in the sense of distributions, where $e^{-x^2/n}$ could be replaced with another sequence of Schwartz functions tending to 1.

(4) Next,

$$\hat{H}[\varphi] = H[\hat{\varphi}] = \int_0^\infty \hat{\varphi}.$$

Using limits we write $e^{-\varepsilon x} H(x) \rightarrow H(x)$ as $\varepsilon \rightarrow 0^+$, and

$$\int_0^\infty e^{-ix\xi} e^{-\varepsilon x} dx = \frac{1}{i\xi + \varepsilon} = \frac{1}{i} \frac{1}{\xi - i\varepsilon} \rightarrow -i(\xi - i0)^{-1} = \hat{H}, \quad (25)$$

where we define

$$(\xi \pm i0)^{-1}[\varphi] = \lim_{\varepsilon \rightarrow 0^+} \int \frac{\varphi(\xi) d\xi}{\xi \pm i\varepsilon}.$$

We can do a similar calculation with another antiderivative of δ . Take the limit $(H - 1)(x) = \lim_{\varepsilon \rightarrow 0^+} e^{\varepsilon x}(H - 1)(x)$ and get

$$-\int_{-\infty}^0 e^{-ix\xi} e^{\varepsilon x} dx = \frac{1}{i\xi - \varepsilon} \rightarrow -i(\xi + i0)^{-1} = (H - 1)^\wedge.$$

Since $(H - 1)^\wedge = \hat{H} - \hat{1}$, and $\hat{1} = 2\pi\delta$, we get

$$\delta(\xi) = \frac{1}{2\pi i} \left((\xi - i0)^{-1} - (\xi + i0)^{-1} \right). \quad (26)$$

We can interpret (26) as saying that the imaginary part of $(\xi - i0)^{-1}$ is $\pi\delta(\xi)$.

EXERCISE 8.

- (1) Check (24) by using the definition of a distributional limit, and the Fourier transform and inversion formulas (18) and (20), and check that you get the same answer if $e^{-x^2/n}$ is replaced by an arbitrary sequence of Schwartz functions tending to 1 in the sense of distributions.
- (2) Use the calculation in (25) and the Fourier inversion formula to find the Fourier transforms of the following functions and distributions on \mathbb{R} , where $a > 0$ is given: $e^{-ax}H(x)$, $e^{ax}H(-x)$, $(x + ia)^{-1}$, $(x - ia)^{-1}$, $(x + i0)^{-1}$, and $(x - i0)^{-1}$.
- (3) Combine the results above, or calculate directly, to find the Fourier transforms of $e^{-a|x|}$ and $1/(a + x^2)$ for $a > 0$.
- (4) Derive (26), in the form

$$(x - i0)^{-1} - (x + i0)^{-1} = 2\pi i\delta(x).$$

without Fourier transforms, instead using the definition $(x \pm i0)^{-1}[\varphi] = \lim_{\varepsilon \rightarrow 0^+} \int \frac{\varphi(x) dx}{x \pm i\varepsilon}$. Find a similar formula for $(x - i0)^{-1} + (x + i0)^{-1}$, but with $\text{pv}(x^{-1})$ (here pv stands for ‘principal value’) in place of $\delta(x)$ and with different constants, using at least one of the following definitions of $\text{pv}(x^{-1})$:

$$\text{pv}(x^{-1})[\varphi] = \lim_{\varepsilon \rightarrow 0^+} \left(\int_{\varepsilon}^{\infty} \frac{\varphi(x)}{x} dx + \int_{-\infty}^{-\varepsilon} \frac{\varphi(x)}{x} dx \right),$$

$$\text{pv}(x^{-1})[\varphi] = - \int_{-\infty}^{\infty} \ln(|x|)\varphi'(x) dx.$$

- (5) Define $L_+ = \ln(x + i0)$ and $L_- = \ln(x - i0)$ in such a way that $L'_+ = (x + i0)^{-1}$, $L'_- = (x - i0)^{-1}$, and L_+ and L_- are each a linear combination of $\ln|x|$ and $H(x)$. How can you change your definition so that they are instead linear combinations of $\ln|x|$ and $1 - H(x)$? (Here $\ln|x|$ should be read as the real logarithm, but for the logarithm of any nonreal number you should choose an appropriate branch.)
- (6) Find the Fourier transforms of $\text{sgn}(x)$ and $\text{pv}(x^{-1})$.

4. The heat equation. Before turning to waves, we use our results on the Fourier transform to solve the heat equation in free space, because this is easier and closely related.

Let $u(x, t)$ denote the density of some quantity, such as heat or chemical concentration, at time t , at a position $x \in \mathbb{R}^d$. The rate of change of the total density in any region Ω is equal to the

total flux F of density into its boundary:

$$\frac{d}{dt} \int_{\Omega} u(y, t) dy = - \int_{\partial\Omega} F \cdot \nu dS,$$

where ν denotes the outward normal vector to $\partial\Omega$. Applying the divergence theorem and differentiating under the integral sign gives

$$\int_{\Omega} \partial_t u(y, t) dy = - \int_{\Omega} \operatorname{div} F dy.$$

Since this must be true for any region Ω , we have

$$\partial_t u(x, t) = - \operatorname{div} F.$$

We now assume that the flux is proportional to the gradient but in the opposite direction, i.e. $F = -a\nabla u$, for some $a > 0$. This leads to the *heat equation*

$$\partial_t u(x, t) = a\Delta u(x, t). \quad (27)$$

Regarding t as a parameter, we look first for solutions which are Schwartz functions in x . Taking the Fourier transform of both sides gives

$$\partial_t \hat{u}(\xi, t) = -a|\xi|^2 \hat{u}(\xi, t),$$

which is solved by

$$\hat{u}(\xi, t) = e^{-a|\xi|^2 t} \hat{u}(\xi, 0) = e^{-a|\xi|^2 t} \hat{u}_0(\xi), \quad (28)$$

where we use u_0 to denote the starting density $u_0(x) = u(x, 0)$. Taking the inverse Fourier transform of both sides gives, for $t > 0$,

$$u(x, t) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-a|\xi|^2 t} \hat{u}_0(\xi) d\xi, \quad (29)$$

or

$$\begin{aligned} u(x, t) &= \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-a|\xi|^2 t} \int e^{-iy\xi} u_0(y) dy d\xi \\ &= \frac{1}{(2\pi)^d} \int \int e^{i(x-y)\xi} e^{-a|\xi|^2 t} d\xi u_0(y) dy, \end{aligned}$$

where in the second equality we used Fubini's theorem to interchange the order of integrals, since $\int e^{-a|\xi|^2 t} d\xi \int |u_0(y)| dy$ converges. By Example 3(1), this simplifies to

$$u(x, t) = \frac{1}{(4\pi at)^{d/2}} \int e^{-\frac{|x-y|^2}{4at}} u_0(y) dy. \quad (30)$$

Hence, for $u_0 \in \mathcal{S}$, there is a unique solution $u(x, t)$ which is a Schwartz function in x for all $t \geq 0$, and it is given by (30) for $t > 0$ and its Fourier transform in x is given by (28) for $t \geq 0$.

These results extend directly to $u_0 \in \mathcal{S}'$, provided the integrals are properly understood. More specifically, (28) means that, for each $t \geq 0$,

$$\hat{u}[\varphi] = \hat{u}_0[e^{-a|\xi|^2 t} \varphi], \quad (31)$$

and (30) means that, for each $t > 0$,

$$u[\varphi] = (4\pi at)^{-d/2} \int u_0 \left[e^{-\frac{|x-\bullet|^2}{4at}} \right] \varphi(x) dx,$$

where $e^{-\frac{|x-\bullet|^2}{4at}}$ means the function $y \mapsto e^{-\frac{|x-y|^2}{4at}}$. We can write more simply

$$u(x, t) = (4\pi at)^{-d/2} u_0 \left[e^{-\frac{|x-\bullet|^2}{4at}} \right].$$

Thus, even if $u_0 \in \mathcal{S}'$, the solution u is a smooth function of x and t for $t > 0$.

If u_0 is bounded above, say $u_0(x) \leq C$ for all x , then by (30) and using the substitution $z = (x - y)/\sqrt{4\pi at}$,

$$u(x, t) \leq \frac{C}{(4\pi at)^{d/2}} \int e^{-\frac{|x-y|^2}{4at}} dy = C \int e^{-\pi z^2} dz = C. \quad (32)$$

A similar calculation shows that if u_0 is bounded below, then $u(x, t)$ obeys the same lower bound.

EXERCISE 9.

- (1) Let $d = 1$ and let $u_0(x) = H(x)$. Write the solution (30) in terms of the error function erf.
- (2) Let $d = 2$ and let $u_0(x_1, x_2) = \text{sgn}(x_1) \text{sgn}(x_2)$. Use the solution to part (1) to write the solution (30) in terms of the error function erf.
- (3) Bonus: Use a program like Desmos or Geogebra or Mathematica (or, if you must, work by hand) to plot u_0 and then u at various times.

If $u_0 \in L^1$, i.e. $\int |u_0| < +\infty$, then $\hat{u}_0(\xi) = \int e^{-ix\xi} u_0(x) dx$ is continuous (use the dominated convergence theorem to show that $\hat{u}_0(\xi_n) \rightarrow \hat{u}_0(\xi)$ when $\xi_n \rightarrow \xi$) and bounded: $|\hat{u}_0(\xi)| \leq \int |u_0|$. Then by (28) we have $\hat{u}(0, \xi) = \hat{u}_0(0)$, i.e.

$$\int u(x, t) dx = \int u_0, \quad (33)$$

for all $t \geq 0$. Moreover, by (29) and using $z = \xi\sqrt{at/\pi}$, we have

$$|u(x, t)| \leq (2\pi)^{-d} \sup |\hat{u}_0| \int e^{-a|\xi|^2 t} d\xi = (4\pi at)^{-d/2} \sup |\hat{u}_0| \leq (4\pi at)^{-d/2} \int |u_0|, \quad (34)$$

In other words, (33) says that the total density is conserved, and (34) says that the maximum density is decaying like $O(t^{-d/2})$.

EXERCISE 10. Show that if $u_0 \in L^1$, then (33) and (34) hold, by using (30) and calculating as in (32).

We can get more precise information on the behavior of $u(x, t)$ for long time under stronger assumptions on u_0 .

EXAMPLE 4. Let $d = 1$ and $u_0(x) = \mathbf{1}_{[-1,1]}$. Then, for any $t > 0$,

$$\begin{aligned} u(x, t) &= \frac{1}{\sqrt{4\pi at}} \int_{-1}^1 e^{-\frac{(x-y)^2}{4at}} dy = \frac{1}{\sqrt{4\pi at}} \int_{-1}^1 \sum_{k=0}^{\infty} \frac{(x-y)^{2k}}{(-4at)^k k!} dy = \frac{1}{\sqrt{4\pi at}} \sum_{k=0}^{\infty} \int_{-1}^1 \frac{(x-y)^{2k}}{(-4at)^k k!} dy \\ &= \frac{1}{\sqrt{4\pi at}} \sum_{k=0}^{\infty} \frac{-(x-y)^{2k+1}}{(2k+1)(-4at)^k k!} \Big|_{y=-1}^{y=1} = \frac{1}{\sqrt{\pi a}} t^{-1/2} - \frac{3x^2+1}{12a\sqrt{\pi a}} t^{-3/2} + \dots \end{aligned}$$

where in the third equality we used the fact that power series may be integrated term-by-term.

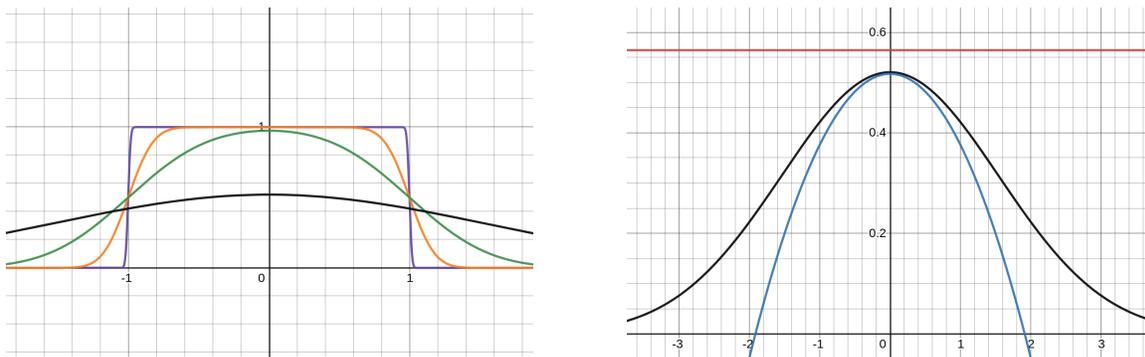


FIGURE 4. On the left, $u(x, t)$ when $u_0(x) = \mathbf{1}_{[-1,1]}$ and $a = 1$, for $t = 0.0001, 0.01, 0.1$, and 1 . On the right, $u(x, 1)$ and its first two approximations, $c_0(x) = \frac{1}{\sqrt{\pi a}}t^{-1/2}$ and $c_1(x) = \frac{1}{\sqrt{\pi a}}t^{-1/2} - \frac{3x^2+1}{12a\sqrt{\pi a}}t^{-3/2}$. The images are from <https://www.desmos.com/calculator/sddvz11iks> and <https://www.desmos.com/calculator/kjaddgflok>.

More generally, under suitable assumptions on u_0 , one can show that

$$\begin{aligned} u(x, t) &= \frac{1}{(4\pi at)^{d/2}} \int \sum_{k=0}^{\infty} \frac{|x-y|^{2k}}{(-4at)^k k!} u_0(y) dy = \frac{1}{(4\pi at)^{d/2}} \sum_{k=0}^{\infty} \int \frac{|x-y|^{2k}}{(-4at)^k k!} u_0(y) dy \\ &= \sum_{k=0}^{\infty} c_k(x) t^{-k-d/2}, \end{aligned} \quad (35)$$

where

$$c_0(x) = (4\pi a)^{-d/2} \int u_0, \quad c_1(x) = -(4\pi a)^{-d/2} (4a)^{-1} \int |x-y|^2 u_0(y) dy, \quad \dots$$

EXERCISE 11.

- (1) Verify (35) under the assumption that u_0 is bounded and compactly supported, i.e. that there exist constants A and B such that $|u_0(x)| \leq A$ for all x and $u_0(x) = 0$ when $|x| \geq B$.
- (2) Prove the weaker statement that $u(x, t) = (4\pi at)^{-d/2} \int u_0 + O(t^{-1-d/2})$, under the weaker assumption that $\int (1 + |y|^2) |u_0(y)| dy$ converges, using $e^s = 1 + \int_0^s e^{s'} ds'$ in place of $e^s = \sum_{k=0}^{\infty} s^k/k!$.
- (3) Bonus: Given $K \geq 0$, prove an expansion up to $O(t^{-K-d/2})$ under a suitable assumption on u , using Taylor's formula with remainder in place of $e^s = 1 + \int_0^s e^{s'} ds'$.

EXERCISE 12. Find functions $c_0(x), c_1(x), \dots$ and constants $k_0 < k_1 < \dots$ such that if $u_0(x) = H(x)$, then $u(x, t) = c_0(x)t^{-k_0} + c_1(x)t^{-k_1} + \dots$.

5. The Schrödinger equation. Though structurally similar to the heat equation, the essential difference being a factor of i , the Schrödinger equation is more difficult to derive and understand the meaning of than the heat equation, requiring either more work or bigger leaps. We will take the latter route.

The *state* of a quantum particle is given by a wavefunction, $u(x, t)$, which is a complex-valued function, called an *amplitude*, at position x and time t . The basic example is the sinusoid

$$u(x, t) = e^{i(k \cdot x - \omega t)} = e^{ik \cdot (x - k\omega t / |k|^2)}.$$

The factorization on the right shows that the wave is traveling in the direction of the *wave vector* k , and so the momentum of such a sinusoid should be proportional to k . Meanwhile the energy is proportional to ω . The *quantization relations* give the constant of proportionality in both cases as *Planck's constant* $\hbar = h/2\pi \approx 1.05 \cdot 10^{-34} \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}$ in SI (metric) units.⁵ In other words

$$E = \hbar\omega, \quad \text{and} \quad \xi = \hbar k. \quad (36)$$

Thus we get

$$u(x, t) = e^{i(\xi \cdot x - Et)/\hbar}. \quad (37)$$

To get a differential equation we bring in the classical energy formula

$$E = \frac{1}{2m}|\xi|^2 + V(x), \quad (38)$$

where m is mass and $V(x)$ is potential energy at x due to the effects of a conservative force. Observe that if u is given by (37), then

$$Eu = i\hbar\partial_t u, \quad \text{and} \quad |\xi|^2 u = -\hbar^2 \Delta u. \quad (39)$$

Multiplying (38) by u and substituting (39) leads to *Schrödinger's equation*:

$$i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u + V(x)u. \quad (40)$$

The point is that we have used (39) to eliminate E and ξ from the equation, which depend on the particular state the particle is in, and left only the constants \hbar and m which are independent of the state. Note also that the first term on the right corresponds to the kinetic energy, and the second to the potential energy. For a free particle $V(x) = 0$ for all x (there is no force) and we get the *free Schrödinger equation*:

$$i\hbar\partial_t u = -\frac{\hbar^2}{2m}\Delta u, \quad \text{or} \quad \partial_t u = i\frac{\hbar}{2m}\Delta u. \quad (41)$$

It is solved by (37) as long as $E = \frac{1}{2m}|\xi|^2$, and more general solutions are obtained from this one by taking superpositions using the Fourier transform. To derive them, we mimic what we did with the heat equation (27), obtaining, with $a = i\frac{\hbar}{2m}$, first $\partial_t \hat{u} = -a|\xi|^2 \hat{u}$ and then

$$\hat{u}(\xi, t) = e^{-a|\xi|^2 t} \hat{u}_0(\xi), \quad u(x, t) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-a|\xi|^2 t} \int e^{-iy\xi} u(y) dy d\xi,$$

with the difference being that for the heat equation a was positive, but for the Schrödinger equation a is pure imaginary. With this difference Fubini's theorem is no longer applicable, so we

⁵Just for fun let us replace the kilogram by the mass of the electron, $9.1 \cdot 10^{-31} \text{ kg}$, one of the meters by the Ångström, 10^{-10} m which is a typical length scale for atomic radii, and the remaining meters per second with a typical electron speed in a hydrogen atom of $2.2 \cdot 10^6 \text{ m/s}$. That gives $\hbar \approx 1/2$.

will proceed more cautiously. We will first assume that $\operatorname{Re} a > 0$, and then at the end extend to the case where a is pure imaginary by taking a limit. We previously deduced

$$u(x, t) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-a|\xi|^2 t} \int e^{-iy\xi} u_0(y) dy d\xi = \frac{1}{(4\pi a t)^{d/2}} \int e^{-\frac{|x-y|^2}{4at}} u_0(y) dy, \quad (42)$$

for $a > 0$. The same equation holds for $\operatorname{Re} a > 0$, since both sides are complex analytic functions of a , provided $a^{d/2}$ means $(\sqrt{a})^d$, with the square root having positive real part. This is called extending an identity by *analytic continuation*.⁶

EXERCISE 13. Above we saw how to prove that if $\operatorname{Re} a > 0$, then

$$\int_{-\infty}^{\infty} e^{ix\xi} e^{-a\xi^2} d\xi = \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{4a}},$$

with the square root taken in the right half plane, by using complex analysis to reduce to the case $a > 0$. Prove the formula in another way, by filling in the details in the following calculation:

$$\int_{-\infty}^{\infty} e^{ix\xi} e^{-a\xi^2} d\xi = e^{-\frac{x^2}{4a}} \int_{-\infty}^{\infty} e^{-a(\xi - \frac{ix}{2a})^2} d\xi = e^{-\frac{x^2}{4a}} \int_{-\infty}^{\infty} e^{-a\xi^2} d\xi = \sqrt{\frac{\pi}{a}} e^{-\frac{x^2}{4a}}.$$

For the first equality, complete the square, for the second either do a contour integral or differentiate under the integral sign with respect to x , and for the third do the usual trick of writing $(\int e^{-a\xi^2} d\xi)^2$ as a double integral and converting to polar coordinates.

Next, for any $u_0 \in \mathcal{S}$ and pure imaginary a , we still have (42) if we replace a by $a + \varepsilon$ and take $\varepsilon \rightarrow 0^+$. Plugging in $a = i\frac{\hbar}{2m}$ gives us the solution to (41) in the form

$$u(x, t) = e^{-i\pi d/4} \left(\frac{m}{2\pi\hbar t} \right)^{d/2} \int e^{\frac{im|x-y|^2}{2\hbar t}} u_0(y) dy, \quad (43)$$

and of course we also still have the Fourier transform formula

$$\hat{u}(\xi, t) = e^{-i\frac{\hbar}{2m}|\xi|^2 t} \hat{u}_0(\xi). \quad (44)$$

We derived all this for $u_0 \in \mathcal{S}$, but the Fourier transform formula (44) still makes sense when $u_0 \in \mathcal{S}'$, just as for the heat equation we interpreted (28) as (31). The formula (43) makes sense when $u_0 \in \mathcal{S}'$ too, if we understand it correctly, but this is trickier, as $u(x, t)$ is again a distribution. When $u_0 \in \mathcal{S}$, the formula (43) defines a solution in \mathcal{S} for all t , and pairing it with a test function φ gives

$$u(x, t)[\varphi] = e^{-i\pi d/4} \left(\frac{m}{2\pi\hbar t} \right)^{d/2} \int \int e^{\frac{im|x-y|^2}{2\hbar t}} u_0(y) dy \varphi(x) dx.$$

Now switching the order of integration is permissible by Fubini's theorem, and allows us to write the expression in a way which makes sense also for $u_0 \in \mathcal{S}'$:

$$u(x, t)[\varphi] = e^{-i\pi d/4} \left(\frac{m}{2\pi\hbar t} \right)^{d/2} u_0 \left[\int e^{\frac{im|x-\bullet|^2}{2\hbar t}} \varphi(x) dx \right], \quad (45)$$

⁶We are using here the fact from complex analysis that if two functions $f(a)$ and $g(a)$ are complex analytic for $\operatorname{Re} a > 0$ and we have $f(a) = g(a)$ when a is real and positive, then we also have $f(a) = g(a)$ whenever $\operatorname{Re} a > 0$. More generally, if two complex analytic functions with a common domain agree on a set with an accumulation point, then they agree throughout their common domain. In other words, the zeros of a nontrivial complex analytic function are isolated: see e.g. Section 3.1 of [Fis], or Section 3.2 of [Ahl].

where on the right hand side, u_0 is being applied to the Schwartz function $y \mapsto \int e^{\frac{im|x-y|^2}{2\hbar t}} \varphi(x) dx$. Another good way to understand (43) when $u_0 \in \mathcal{S}'$ is as the limit in the sense of distributions of

$$\begin{aligned} u(x, t) &= \lim_{\varepsilon \rightarrow 0^+} e^{-i\pi d/4} \left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} \int e^{\frac{i(m+i\varepsilon)|x-y|^2}{2\hbar t}} u_0(y) dy \\ &= \lim_{\varepsilon \rightarrow 0^+} e^{-i\pi d/4} \left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} u_0 \left[e^{\frac{i(m+i\varepsilon)|x-\bullet|^2}{2\hbar t}} \right]; \end{aligned} \quad (46)$$

this corresponds to the calculation we used to derive the formula.

EXERCISE 14. Show that the two ways of understanding the formula (43) described above, namely (45) and (46), are equivalent, by using the following steps:

$$\begin{aligned} (45) &= e^{-i\pi d/4} u_0 \left[\lim_{\varepsilon \rightarrow 0^+} \left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} \int e^{\frac{i(m+i\varepsilon)|x-\bullet|^2}{2\hbar t}} \varphi(x) dx \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} e^{-i\pi d/4} u_0 \left[\left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} \int e^{\frac{i(m+i\varepsilon)|x-\bullet|^2}{2\hbar t}} \varphi(x) dx \right] \\ &= \lim_{\varepsilon \rightarrow 0^+} e^{-i\pi d/4} \left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} \int u_0 \left[e^{\frac{i(m+i\varepsilon)|x-\bullet|^2}{2\hbar t}} \right] \varphi(x) dx \\ &= \int \lim_{\varepsilon \rightarrow 0^+} e^{-i\pi d/4} \left(\frac{m+i\varepsilon}{2\pi\hbar t} \right)^{d/2} u_0 \left[e^{\frac{i(m+i\varepsilon)|x-\bullet|^2}{2\hbar t}} \right] \varphi(x) dx. \end{aligned}$$

Justify the first equality using the dominated convergence theorem. For the second, use the following *continuity property*: if $u_0 \in \mathcal{S}'$ then there are constants C , M , and N such that $|u_0[\varphi]| \leq C \sup |(1+x^2)^M \partial^\alpha \varphi(x)|$, where the supremum is taken over partial derivatives ∂^α of order $\leq M$ and over $x \in \mathbb{R}^d$. For the third, first prove it when $u_0 \in \mathcal{S}$, and then take a limit in the sense of distributions to prove it for $u_0 \in \mathcal{S}'$. For the fourth, use the continuity property again and the dominated convergence theorem. For more on the continuity property see the references in the ‘Further discussion and references’ section below.

EXAMPLE 5.

(1) Let $u_0(x) = e^{ix\xi_0/\hbar}$. Then

$$\hat{u}_0[\varphi] = \int e^{ix\xi_0/\hbar} \hat{\varphi}(x) dx = \int e^{ix\xi_0/\hbar} \int e^{-ixy} \varphi(y) dy dx = (2\pi)^d \varphi(\xi_0/\hbar),$$

so $\hat{u}_0 = (2\pi)^d \delta_{\xi_0/\hbar}$, and by (44) we have

$$\hat{u}(\xi, t) = e^{-i\frac{\hbar}{2m}|\xi|^2 t} \hat{u}_0 = (2\pi)^d e^{-i\frac{1}{2m\hbar}|\xi_0|^2 t} \delta_{\xi_0/\hbar}.$$

That gives

$$u(x, t) = \frac{1}{(2\pi)^d} \int e^{ix\xi} \hat{u}(\xi, t) d\xi = e^{ix\xi_0/\hbar} e^{-i\frac{1}{2m\hbar}|\xi_0|^2 t},$$

which recovers (37), with ξ_0 in place of ξ .

(2) Let $d = 1$, and let $u_0(x) = e^{-x^2/2L^2}$. Then $\hat{u}_0(\xi) = L\sqrt{2\pi}e^{-\xi^2 L^2/2}$, and, working in units such that $\hbar/m = 1$, we have

$$\hat{u}(\xi, t) = e^{-i\xi^2 t/2} \hat{u}_0(\xi) = L\sqrt{2\pi}e^{-(L^2+it)\xi^2/2},$$

and so

$$u(x, t) = \frac{1}{2\pi} \int e^{ix\xi} \hat{u}(\xi, t) d\xi = \frac{L}{\sqrt{2\pi}} \int e^{ix\xi} e^{(L^2+it)\xi^2/2} d\xi = \frac{L}{\sqrt{L^2+it}} e^{-x^2/(2L^2+it)}.$$

If we want to remove the assumption that the units are such that $\hbar/m = 1$, it suffices to replace t by $t\hbar/m$ in the formula. That gives

$$u(x, t) = \sqrt{\frac{mL}{mL^2 + i\hbar t}} e^{-mx^2/(2mL^2 + i\hbar t)}. \quad (47)$$

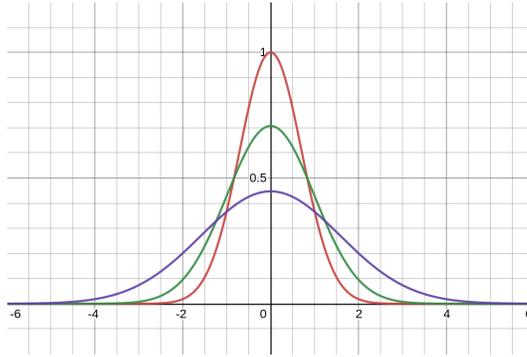


FIGURE 5. $|u(x, t)|^2$ when $u_0(x) = e^{-x^2/2}$ and $L = m = \hbar = 1$, for $t = 0, 1$, and 2 . The image is from <https://www.desmos.com/calculator/vfajqntuuo>.

The probability that the particle whose state is given by $u(x, t)$ is found at position x at time t is given by

$$P(x, t) = |u(x, t)|^2 / \int_{\mathbb{R}^d} |u(y, t)|^2 dy,$$

when the integral in the denominator converges. Since u itself is complex valued, one often graphs $P(x, t)$ or $|u(x, t)|^2$ for this reason. In the case of (47), we have

$$|u(x, t)|^2 = \frac{mL}{\sqrt{m^2L^4 + \hbar^2t^2}} e^{-m^2L^2x^2/(m^2L^4 + \hbar^2t^2)}.$$

A solution given by (47) is called a *Gaussian wave packet* and the one derived above stays put while gradually dispersing. There also exist traveling Gaussian wave packets.

More generally, working for simplicity in units such that $m = \hbar = 1$, if $u(x, t)$ is any solution to (41), then so is

$$\tilde{u}(x, t) = e^{ivx} e^{-it|v|^2/2} u(x - vt, t), \quad (48)$$

for any constant $v \in \mathbb{R}^d$. The mapping $u \mapsto \tilde{u}$ above is called the *Galilean transformation*. It is the quantum version of the classical Galilean transformation $x(t) \mapsto x(t) - vt$, which takes the trajectory of a particle in one reference frame to its trajectory in a second reference frame moving at velocity v with respect to the first.

EXERCISE 15. Construct a *traveling Gaussian wave packet* in at least one of the following two ways:

- (1) Apply the Galilean transformation to the standing Gaussian wave packet solution (47) for some choice of v .
- (2) Use an initial condition such that $\hat{u}_0(\xi)$ is a constant multiple of $e^{-|\xi-\xi_0|^2 L^2/2}$ for some choice of ξ_0 .

Make a plot of the resulting modulus squared of the solution as in Figure 5. Bonus: what is the relationship between the two constructions?

EXERCISE 16. Use the chain rule and product rule to check that if $u(x, t)$ is any solution to the Schrödinger equation (41), then so is $\tilde{u}(x, t)$ given by (48). Alternatively,⁷ check this by replacing x by $x - vt$ in the solution formula (43) and writing $e^{\frac{i|x-vt-y|^2}{2t}} = e^{\frac{i|x-y|^2}{2t}} e^{it|v|^2/2} e^{-ivx} e^{ivy}$. This alternative approach is also a good way to do the next two exercises.

EXERCISE 17. Derive (48) using the following steps.

- (1) For each ξ , let $S_\xi(x, t) = e^{ix\xi} e^{-i|\xi|^2 t/2}$.
- (2) Multiply and divide to write $S_\xi(x, t) = e^{i(x-vt)\xi} e^{-i|\xi|^2 t/2} e^{ivt\xi}$. This writes $S_\xi(x, t)$ as a function of $x - vt$.
- (3) Complete the square in the last two factors $e^{-i|\xi|^2 t/2} e^{ivt\xi} = \exp(-i\frac{t}{2}(|\xi|^2 - 2v\xi))$ to write these two factors as a function of $\xi - v$.
- (4) Plug the result of (3) into (2), and also plug in $e^{i(x-vt)\xi} = e^{i(x-vt)(\xi-v)} e^{-ivt^2} e^{ixv}$ to write $S_\xi(x, t)$ in terms of $S_{\xi-v}(x - vt)$.
- (5) The result of (4) shows that applying the Galilean transformation (48) to a sinusoidal solution $S_\xi(x, t)$, for any ξ , gives another sinusoidal solution.
- (6) Since general solutions are superpositions of sinusoidal ones by (44), it follows that if $u(x, t)$ is any solution, then so is $\tilde{u}(x, t)$ given by (48).

EXERCISE 18. Give a more general formula for the Galilean transformation without the assumption that $\hbar = m = 1$, and/or give a formula where $u(x - vt, t)$ is replaced by $u(x - x_0 - vt, t)$ for some $x_0 \in \mathbb{R}^d$; the latter corresponds to the classical Galilean transformation $x(t) \mapsto x(t) - x_0 - vt$. Bonus: What happens if t is replaced by $t - t_0$ for some $t_0 \in \mathbb{R}$?

An important consequence of the Fourier transform formula (44) is that

$$|\hat{u}(\xi, t)| = |\hat{u}_0(\xi)| \tag{49}$$

for all t . The physical interpretation of (49) is as follows: the relative probability that the particle whose state is given by $u(x, t)$ is found at position x at time t is proportional to $|u(x, t)|^2$, and the relative probability that it is at momentum ξ/\hbar is proportional to $|\hat{u}(\xi/\hbar, t)|$. Thus (49) says the momentum distribution of the particle is constant in time; this is the quantum version of the basic Newtonian principle that momentum is unchanged in the absence of forces.

The equation (49) also has important implications for the position distribution $|u(x, t)|^2$, the primary one being that

$$\int |u(x, t)|^2 dx = \int |u_0(x)|^2 dx, \tag{50}$$

⁷Thanks to Scott Kenning for suggesting this.

for all t . To derive (50), we use the fact that, by the definition of the Fourier transform, we have $\hat{f}(\xi) = \tilde{f}(-\xi)$, and by Fourier inversion (Theorem 3), we have $\hat{f}(x) = (2\pi)^d f(-x)$. Together these give $\hat{\hat{f}} = (2\pi)^d \tilde{f}$. Recalling also from (21) that $\int f \hat{g} = \int \hat{f} g$, we obtain

$$\int f \bar{g} = (2\pi)^d \int f \hat{\hat{g}} = (2\pi)^{-d} \int \hat{f} \bar{\hat{g}},$$

and in particular

$$\int |f|^2 = (2\pi)^{-d} \int |\hat{f}|^2. \quad (51)$$

The above two identities are called the *Parseval* or *Plancherel* identities. Then (50) follows from (49) and (51). Thus the Fourier transform (by (51)), and also evolution under the free Schrödinger equation (by (50)), both preserve the L^2 norm $\|u\|_{L^2(\mathbb{R}^d)} = \sqrt{\int_{\mathbb{R}^d} |u|^2}$.

An $O(t^{-d/2})$ bound holds for the maximum of u , analogous to that obtained previously for the heat equation in (34). Working from (43), we have

$$|u(x, t)| \leq \left| \frac{m}{2\pi\hbar t} \right|^{d/2} \int |u_0| = O(t^{-d/2}).$$

EXERCISE 19. Derive a series expansion for $u(x, t)$ of the form

$$u(x, t) = \sum_{k=0}^{\infty} c_k(x) t^{-k-d/2},$$

analogous to the one derived for the heat equation in (35), either in the case that $d = 1$ and $u_0(x) = \mathbf{1}_{[-1,1]}(x)$, or in a more general case.

Free Schrödinger evolution also preserves regularity in the very important *Sobolev* sense, and this is our next topic.

6. Sobolev Spaces.

DEFINITION 4. Let k be a nonnegative integer. The *Sobolev space of order k on \mathbb{R}^d* , denoted $H^k(\mathbb{R}^d)$, is the set of all tempered distributions u such that u and all its derivatives up to order k are in $L^2(\mathbb{R}^d)$.

Using $\widehat{u_{x_j}} = i\xi_j \hat{u}$ and (49) shows that if $u(x, t)$ solves the free Schrödinger equation

$$i\hbar\partial_t u(x, t) = -\frac{\hbar^2}{2m}\Delta u(x, t), \quad u(x, 0) = u_0(x),$$

where $u_0 \in H^k$ for a given k , then $u(x, t) \in H^k$ for all t .

More generally, we define:

DEFINITION 5. Let s be a real number. The *Sobolev space of order s on \mathbb{R}^d* , denoted $H^s(\mathbb{R}^d)$, is the set of all tempered distributions u such that $(1 + |\xi|^2)^{s/2} \hat{u}(\xi)$ is in $L^2(\mathbb{R}^d)$.

Then (49) shows that, for solutions to the Schrödinger equation $u_0 \in H^s$ for a given s if and only if $u(x, t)$ is for all t . In other words, solutions to the Schrödinger equation neither gain nor lose Sobolev regularity. A calculation based on (51) shows that the definitions agree when s is a nonnegative integer:

EXERCISE 20. Check that if $s = k$ is a nonnegative integer then Definitions 4 and 5 agree. For $n \geq 2$ you will need some multinomial algebra, including the fact that any monomial in ξ of order m is less than or equal to $|\xi|^m$

EXERCISE 21. Prove that if the initial condition $u_0(x)$ of a solution to the heat equation is in $H^s(\mathbb{R}^d)$ for *some* real s , then for all $t > 0$ the solution $u(x, t)$ is in $H^{s'}(\mathbb{R}^d)$ for *all* real s' .

EXAMPLE 6.

- (1) Since $\hat{\delta}_0 = 1$, and $\int_{\mathbb{R}^d} (1 + |\xi|^2)^s d\xi$, which is proportional⁸ to $\int_0^\infty (1 + r^2)^s r^{d-1} dr$, converges if and only if $s < -d/2$, we see that $\delta_0 \in H^s(\mathbb{R}^d)$ if and only if $s < -d/2$.
- (2) Let I be a bounded interval in \mathbb{R} , and let $\mathbf{1}_I$ be the characteristic function of I . Using the fact that there is a constant C such that for all ξ we have

$$|\hat{\mathbf{1}}_I(\xi)| \leq C(1 + |\xi|^2)^{-1/2},$$

which follows from the bound $|\hat{\mathbf{1}}_I(\xi)| \leq \int_a^b 1 dx = (b - a)$ for $|\xi| \leq 1$ and the bounds $|\hat{\mathbf{1}}_I(\xi)| \leq 2|\xi|^{-1}$ and $|\xi|^2 \geq \frac{1}{2}(1 + |\xi|^2)$ for $|\xi| \geq 1$, we see that $\mathbf{1}_I \in H^s(\mathbb{R})$ for all $s < 1/2$. If we stick to integer order spaces, the best we can say is $\mathbf{1}_I \in H^0 = L^2$.

- (3) Define $u: \mathbb{R} \rightarrow \mathbb{R}$ by $u(x) = 1 - |x|$ for $|x| < 1$ and $u(x) = 0$ otherwise. Then u' is a linear combination of characteristic functions of intervals, so $u \in H^s(\mathbb{R})$ for $s < 3/2$.

EXERCISE 22. Prove that if $u \in H^s(\mathbb{R}^d)$, then $\partial_{x_j} u \in H^{s-1}(\mathbb{R}^d)$.

EXERCISE 23. Prove that if $u' \in H^s(\mathbb{R})$ and $u \in H^r(\mathbb{R})$, then $u' \in H^{s-1}(\mathbb{R})$, regardless of the values of s and r . Bonus: Formulate and prove a version of this result for \mathbb{R}^d .

EXERCISE 24. Define $u: \mathbb{R} \rightarrow \mathbb{R}$ by $u(x) = 1 - |x|$ for $|x| < 1$ and $u(x) = 0$ otherwise. Fill in the details of Example 6(3) and show that if $s < 3/2$, then $u \in H^s(\mathbb{R})$.

EXERCISE 25. Let $a > 0$. Use the result of Exercise 8(3) to find the values of s for which the functions $e^{-a|x|}$ and $1/(a + x^2)$ belong to $H^s(\mathbb{R})$.

EXERCISE 26. Functions of the form $u(x) = e^{-|x|^\alpha}$, where $\alpha > 0$, have a *cusp singularity* at the origin. They are important in quantum mechanics because the least energy wave function of an electron bound by an atomic nucleus has this form, and more general wave functions of electrons in molecules have a similar form (with cusps at atomic nuclei, and exponential decay). See Chapters 8 and 12 of [Ham] for more on this.

⁸This can be checked using polar coordinates $x = r\omega$, where $r = |x|$, and $\omega = x/|x|$ ranges over the unit sphere \mathbb{S}^{d-1} , and where ω is in turn written in terms of some local coordinates on \mathbb{S}^{d-1} (e.g. angles the vector $x/|x|$ makes with certain coordinate axes/planes; if $d = 2$ we use $\omega = (\cos \theta, \sin \theta)$, and if $d = 3$ we use $(\cos \theta \sin \varphi, \sin \theta \sin \varphi, \cos \varphi)$). By the change of variables formula, that gives $\int_{\mathbb{R}^d} f(x) dx = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(r\omega) r^{d-1} dS dr$, or, if f is independent of ω , $\int_{\mathbb{R}^d} f(|x|) dx = A_{d-1} \int_0^\infty f(r) r^{d-1} dr$, where dS is the surface element on \mathbb{S}^{d-1} and $A_d = \int_{\mathbb{S}^{d-1}} dS$ is the length/area/volume of \mathbb{S}^{d-1} . See e.g. [Fol, Section 2.7] for more.

- (1) Use the polar coordinates formula $\partial_{x_j} = \frac{x_j}{r} \partial_r$, where $r = |x|$, to show that any partial derivative of order k of u is bounded in a neighborhood of the origin by a multiple of $|x|^{\alpha-k}u(x)$
- (2) Conclude that $u \in H^k(\mathbb{R}^d)$ for any nonnegative integer k less than $\alpha + \frac{d}{2}$.

There is an important relationship between regularity in the Sobolev sense and regularity in the usual C^k sense:

THEOREM 4. *If $u \in H^s(\mathbb{R}^d)$ and $2s > 2k + d$, where k is a nonnegative integer, then $u \in C^k(\mathbb{R}^d)$.*

Proof. By Fourier inversion,

$$u(x) = \frac{1}{(2\pi)^d} \int e^{ix\xi} \hat{u}(\xi) d\xi, \quad (52)$$

we know that $u \in C^0$ if $\hat{u} \in L^1$. To check that $\hat{u} \in L^1$, note that, by the Cauchy–Schwarz inequality⁹,

$$\left(\int |\hat{u}| \right)^2 \leq \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi \int (1 + |\xi|^2)^{-s} d\xi,$$

and the two integrals on the right converge (the first because $u \in H^s$, the second because $s > d/2$). If $k > 0$, then differentiate (52) to show that partial derivatives up to order k of u are continuous as well. \square

7. The wave equation. We now return to the wave equation,

$$\partial_t^2 u(x, t) - c^2 \Delta u(x, t) = 0, \quad u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x), \quad (53)$$

where $c > 0$ is a constant, $x \in \mathbb{R}^d$, $t \in \mathbb{R}$. We derived and studied the simpler case $d = 1$ of this above in Section 1. A similar derivation justifies the use of this equation when $d = 2$ to describe the vibrations of a membrane. When $d = 3$ it describes acoustic waves, elastic waves, and electromagnetic waves.

To solve it, we use the same Fourier transform approach as for the heat and Schrödinger equations to reduce to a constant coefficient ordinary differential equation with respect to t :

$$\partial_t^2 \hat{u}(\xi, t) + c^2 |\xi|^2 \hat{u}(\xi, t) = 0.$$

The solutions are linear combinations of sine and cosine functions:

$$\hat{u}(\xi, t) = A(\xi) \cos(c|\xi|t) + B(\xi) \sin(c|\xi|t).$$

Using the initial conditions gives

$$\hat{u}(\xi, t) = \cos(c|\xi|t) \hat{u}_0(\xi) + t \operatorname{sinc}(c|\xi|t) \hat{u}_1(\xi), \quad (54)$$

where $\operatorname{sinc}(s) = \sin(s)/s$.

⁹The Cauchy–Schwarz inequality says that in any inner product space we have $|\langle f, g \rangle| \leq \|f\| \|g\|$, where $\|f\| = \sqrt{\langle f, f \rangle}$. In our setting $\langle f, g \rangle = \int f \bar{g}$. To prove the inequality, observe that for any c we have $0 \leq \|f - cg\|^2 = \|f\|^2 - c \langle g, f \rangle - \overline{c \langle g, f \rangle} + |c|^2 \|g\|^2 = \|f\|^2 - \frac{|\langle f, g \rangle|^2}{\|g\|^2} + \left| \frac{\langle f, g \rangle}{\|g\|} - c \|g\| \right|^2$, where in the last step we completed the square. Set $c = \frac{\langle f, g \rangle}{\|g\|^2}$ to minimize the right hand side (i.e. choose c to make cg approximate f as closely as possible) and conclude that $0 \leq \|f\|^2 - \frac{|\langle f, g \rangle|^2}{\|g\|^2}$ which is equivalent to $|\langle f, g \rangle| \leq \|f\| \|g\|$.

The formula (54) shows that if u_0 and u_1 are Schwartz functions, then there is a unique solution $u(x, t)$ which is a Schwartz function for all t , and its Fourier transform is given by (54). The same formula (54) gives, for any tempered distributions u_0 and u_1 , the solution $u(x, t)$ which is a tempered distribution for all t .

EXERCISE 27. Prove that if u_0 and u_1 are Schwartz functions, then $\hat{u}(\xi, t)$ defined by (54) is in $C^\infty(\mathbb{R}^{d+1})$, and the function $\xi \mapsto \hat{u}(\xi, t)$ is a Schwartz function for each t . Prove that if $u_0 \in H^s(\mathbb{R}^d)$, and $u_1 \in H^{s-1}(\mathbb{R}^d)$ for some s , then $x \mapsto u(x, t) \in H^s$ for all t . *Hint:* Use Taylor series to show that there is no singularity at 0.

For Schrödinger evolution we had $\hat{u}(\xi, t) = e^{-i\frac{\hbar}{2m}t}\hat{u}_0(\xi)$ and hence $|\hat{u}(\xi, t)| = |\hat{u}_0(\xi)|$ for all ξ and t , from which it followed that $\int |u(x, t)|^2 dx = \int |u_0|^2$ and $\int |\partial_x^\alpha u(x, t)|^2 dx = \int |\partial_x^\alpha u_0|^2$ for all t , as long as the initial condition u_0 was such that the integrals on the right hand side converge. Hence $u_0 \in H^s$ for any s implied $u(x, t) \in H^s$ for all t , for the same s .

To get analogs of these results for the wave equation, we use (54) and the Pythagorean identity $\cos^2 + \sin^2 = 1$. We have

$$\begin{aligned} c^2|\xi|^2|\hat{u}(\xi, t)|^2 &= c^2|\xi|^2 \cos^2 |\hat{u}_0|^2 + \sin^2 |\hat{u}_1|^2 + 2c|\xi| \sin \cos \operatorname{Re} \hat{u}_0 \overline{\hat{u}_1}, \\ \partial_t \hat{u}(\xi, t)|^2 &= c^2|\xi|^2 \sin^2 |\hat{u}_0|^2 + \cos^2 |\hat{u}_1|^2 - 2c|\xi| \sin \cos \operatorname{Re} \hat{u}_0 \overline{\hat{u}_1}, \end{aligned}$$

where we have suppressed the arguments of the functions \sin , \cos , \hat{u}_0 , \hat{u}_1 on the right for brevity. Adding the equations gives

$$|\partial_t \hat{u}(\xi, t)|^2 + c^2|\xi|^2|\hat{u}(\xi, t)|^2 = |\hat{u}_1(\xi)|^2 + c^2|\xi|^2|\hat{u}_0(\xi)|^2. \quad (55)$$

which is the analog of (49). Integrating and using Plancherel's theorem (51) gives

$$\int \left(|\partial_t u(x, t)|^2 + c^2 |\nabla u(x, t)|^2 \right) dx = \int \left(|u_1(x)|^2 + c^2 |\nabla u_0(x)|^2 \right) dx. \quad (56)$$

This is called *energy conservation* for the wave equation.

To understand the interpretation of this quantity as energy, we return to the vibrating string problem from Section 1. The kinetic energy and potential energies of the segment of string from a to b are

$$\text{kinetic energy} = \frac{1}{2} \int_a^b \rho \partial_t u(x, t)^2 dx, \quad \text{potential energy} = \int_a^b \tau (\sqrt{\partial_x u(x, t)^2 + 1} - 1) dx.$$

The first is obtained by calculating half the mass times the speed squared, and the second by taking the force of tension times the distance over which it is applied, namely the difference between the length of the curve $x \mapsto (x, u(x, t))$ and the length of the curve $x \mapsto (x, 0)$. For small vibrations, the binomial series $(1 + u_x^2)^s = 1 + su_x^2 + \dots$ gives $\sqrt{\partial_x u(x, t)^2 + 1} \approx 1 + \frac{1}{2} \partial_x u(x, t)^2$ and hence a potential energy of $\int_a^b \tau \partial_x u(x, t)^2 dx$. Thus the total energy of the string is

$$\frac{1}{2} \int \left(\rho \partial_t u(x, t)^2 + \tau \partial_x u(x, t)^2 \right) dx. \quad (57)$$

Recalling that $c = \sqrt{\tau/\rho}$, we see that (57) equals the left side of (56) up to a constant factor.

EXERCISE 28. A more general version of the wave equation (53) to which the methods above apply is

$$\partial_t^2 u(x, t) - c^2 \Delta u(x, t) + a \partial_t u(x, t) + ku(x, t) = 0, \quad u(x, 0) = u_0(x), \quad \partial_t u(x, 0) = u_1(x),$$

where $a \geq 0$ and $k \geq 0$ are constants. In the setting of a vibrating string or membrane, the $a \partial_t u$ term represents a viscous damping force such as air resistance, and the ku term represents an elastic restoring force obeying Hooke's law. Above we had $a = k = 0$. Allow one or both of these constants to be nonzero and derive analogs of (54), (55), and/or (56). Note that when $a = 0$ energy is conserved, but when $a > 0$ it decays. *Hint:* Show that if $\hat{u} = e^{-at/2}(C_1(\xi) \cos(\beta t) + C_2(\xi) \sin(\beta t))$ with β real, then $e^{at}(|\partial_t \hat{u} + \frac{a}{2} \hat{u}|^2 + \beta^2 |\hat{u}|^2)$ is independent of t .¹⁰ If β is imaginary, and it is perhaps too hard to find a good analog of this, then find instead an analog of (56) by starting with the $a = 0$ version and showing that in the more general case the derivative of this energy obeys $\partial_t E = -2a \int |\partial_t u|^2 dx \leq 0$.

Our next step in understanding the solution to the wave equation is to simplify

$$u(x, t) = \frac{1}{(2\pi)^d} \int e^{ix\xi} \left(\cos(c|\xi|t) \hat{u}_0(\xi) + \frac{\sin(c|\xi|t)}{c|\xi|} \hat{u}_1(\xi) \right) d\xi. \quad (58)$$

This is more complicated than the corresponding problem was for the heat and Schrödinger equations, and we proceed in steps. We start by replacing the trigonometric functions with an exponential one, more specifically

$$u(x, y) = \frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-y|\xi|} \hat{u}_0(\xi) d\xi, \quad (59)$$

where $\operatorname{Re} y > 0$; later we will take the limit as y approaches the imaginary axis and use the fact that $e^{\pm ic|\xi|t} = \cos(c|\xi|t) \pm i \sin(c|\xi|t)$. This analogous to how, when solving the Schrödinger equation above, we considered $\frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-a|\xi|^2} \hat{u}_0(\xi) d\xi$ for $\operatorname{Re} a > 0$ and then took the limit as a approached the imaginary axis.

The function $u(x, y)$ in (59) also has its own interest. For $y > 0$, it is the bounded solution to Laplace's equation in the half space $(x, y) \in \mathbb{R}^d \times (0, \infty)$, where

$$\sum_{j=1}^d \partial_{x_j}^2 u(x, y) + \partial_y^2 u(x, y) = 0 \text{ for } y > 0, \quad u(x, 0) = u_0(x). \quad (60)$$

Indeed, taking the Fourier transform of Laplace's equation gives

$$(\partial_y^2 - |\xi|^2) \hat{u}(\xi, y) = 0, \quad \text{which is solved by} \quad \hat{u}(\xi, y) = A(\xi) e^{-|\xi|y} + B(\xi) e^{|\xi|y},$$

and for bounded solutions we require $B = 0$. When $d = 2$, $u(x, y)$ is the electrostatic potential caused by a charge distribution $u_0(x)$ on a flat surface.

From (59) and using Fubini's theorem to switch the order of integration we have

$$u(x, y) = \frac{1}{(2\pi)^d} \int \int e^{i(x-x')\xi} e^{-y|\xi|} d\xi u_0(x') dx'. \quad (61)$$

¹⁰Thanks to Gavin Glenn for pointing this out.

Now notice that if the exponential with $-|\xi|$ were replaced by an exponential with $-|\xi|^2$, we would have a Gaussian Fourier transform, which we know how to evaluate from Example 3 (1) or from Exercise 13. To bring about such a replacement we use two diabolical tricks: we have

$$\int_{-\infty}^{\infty} \frac{e^{iy|\xi|s}}{s^2 + 1} ds = \pi e^{-y|\xi|}, \quad (62)$$

by contour integration (see Figure 6), and

$$\int_0^{\infty} e^{-u} e^{-us^2} du = \frac{1}{s^2 + 1}, \quad (63)$$

by direct calculation.

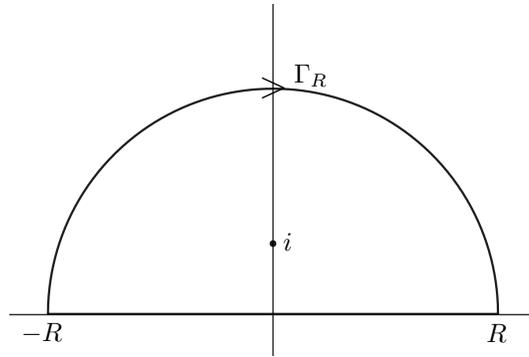


FIGURE 6. By the residue theorem (see e.g. Example 4 from Section 2.6 of [Fis]),

$$\int_{-\infty}^{\infty} \frac{e^{iy|\xi|s}}{s^2 + 1} ds = \lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{e^{iy|\xi|s}}{s^2 + 1} ds + 2\pi i \operatorname{Res} \left(\frac{e^{iy|\xi|s}}{(s+i)(s-i)}; i \right),$$

where

$$\left| \int_{\Gamma_R} \frac{e^{iy|\xi|s}}{s^2 + 1} ds \right| = \left| - \int_0^{\pi} \frac{e^{iy|\xi|Re^{i\theta}}}{R^2 e^{2i\theta} + 1} R i e^{i\theta} d\theta \right| \leq \int_0^{\pi} \frac{1}{R^2 - 1} R d\theta \xrightarrow{R \rightarrow \infty} 0,$$

and

$$2\pi i \operatorname{Res} \left(\frac{e^{iy|\xi|s}}{(s+i)(s-i)}; i \right) = 2\pi i \frac{e^{-y|\xi|}}{2i} = \pi e^{-y|\xi|}.$$

EXERCISE 29. Prove (62) without contour integration, by computing the Fourier transform of the function $u: \mathbb{R} \rightarrow \mathbb{R}$ given by $u(x) = e^{-|x|}$ and using Fourier inversion. (See also Exercise 8 (3).)

Combining (62) and (63) gives

$$\int_{\mathbb{R}^d} e^{i(x-x')\xi} e^{-y|\xi|} d\xi = \frac{1}{\pi} \int_{\mathbb{R}^d} e^{i(x-x')\xi} \int_{-\infty}^{\infty} e^{iy|\xi|s} \int_0^{\infty} e^{-u} e^{-us^2} du ds d\xi.$$

Switching the order of the inner two integrals, and plugging in the Fourier transform formula $\int_{-\infty}^{\infty} e^{iy|\xi|s} e^{-us^2} ds = \sqrt{\frac{\pi}{u}} e^{-y^2|\xi|^2/4u}$, gives

$$\int_{\mathbb{R}^d} e^{i(x-x')\xi} e^{-y|\xi|} d\xi = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^d} e^{i(x-x')\xi} \int_0^{\infty} \frac{1}{\sqrt{u}} e^{-y^2|\xi|^2/4u} e^{-u} du d\xi.$$

Switching the order of the remaining integrals, and plugging in the Fourier transform formula $\int_{\mathbb{R}^d} e^{i(x-x')\xi} e^{-y^2|\xi|^2/4u} d\xi = \frac{(4u\pi)^{d/2}}{y^d} e^{-u|x-x'|^2/y^2}$, gives

$$\int_{\mathbb{R}^d} e^{i(x-x')\xi} e^{-y|\xi|} d\xi = \frac{2^d \pi^{\frac{d-1}{2}}}{y^d} \int_0^\infty u^{\frac{d-1}{2}} e^{-u|x-x'|^2/y^2} e^{-u} du.$$

Combining the exponentials on the right hand side and substituting $u \mapsto u/(1 + |x - x'|^2/y^2) = y^2 u/(y^2 + |x - x'|^2)$ gives

$$\int_{\mathbb{R}^d} e^{i(x-x')\xi} e^{-y|\xi|} d\xi = \frac{2^d \pi^{\frac{d-1}{2}} y}{(y^2 + |x - x'|^2)^{\frac{d+1}{2}}} \int_0^\infty u^{\frac{d-1}{2}} e^{-u} du.$$

The last remaining integral can be written in terms of the gamma function (or see Exercise 31 below) but the main thing is the dependence on the variables x, x', y . Finally, plugging back into (61) gives

$$u(x, y) = C_d y \int_{\mathbb{R}^d} \frac{u_0(x') dx'}{(y^2 + |x - x'|^2)^{\frac{d+1}{2}}}, \quad C_d = \pi^{-\frac{d+1}{2}} \int_0^\infty u^{\frac{d-1}{2}} e^{-u} du. \quad (64)$$

EXAMPLE 7. Let $d = 1$ and $u_0(x) = \mathbf{1}_{[-1,1]}$. Then

$$u(x, y) = \frac{y}{\pi} \int_{-1}^1 \frac{dx'}{y^2 + (x - x')^2} = \frac{1}{\pi} \arctan\left(\frac{x+1}{y}\right) - \frac{1}{\pi} \arctan\left(\frac{x-1}{y}\right).$$

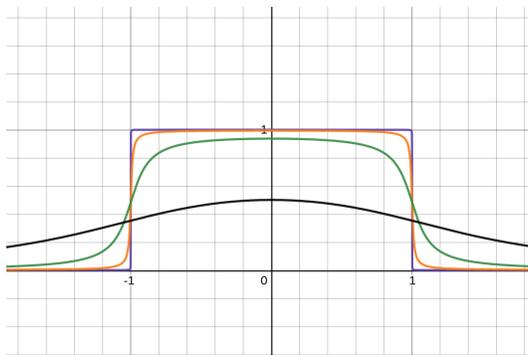


FIGURE 7. The solution $u(x, y)$ to (60) when $u_0(x) = \mathbf{1}_{[-1,1]}$, for $y = 0.0001, 0.01, 0.1$, and 1 . The image is from <https://www.desmos.com/calculator/zpg4xcffsv>.

EXERCISE 30. Let $a > 0$. Use the above calculation to find the values of s for which the function $e^{-a|x|}$ is in $H^s(\mathbb{R}^d)$. (Compare with Exercises 25 and 26.)

EXERCISE 31. Prove that

$$\int_0^\infty u^{\frac{d-1}{2}} e^{-u} du = \begin{cases} \frac{d-1}{2}! & \text{if } d \text{ is odd,} \\ \sqrt{\pi} \left(\frac{d-1}{2}\right) \left(\frac{d-3}{2}\right) \cdots \left(\frac{1}{2}\right) & \text{if } d \text{ is even.} \end{cases}$$

Hint: Use integration by parts to reduce the value of d . When $d = 1$ the integral is easy, and when $d = 0$ use the substitution $u = r^2$ (and note that the formula must be modified when $d = 0$).

EXERCISE 32. The solution to Laplace's equation (60) given by (59) and (64) has many analogous features to the solution to the heat equation (27) given by (28) and (30). Explore some of them: for instance show if u_0 obeys an upper or lower bound then u obeys the same, write the solution for $u_0(x) = H(x)$ in terms of arctan and plot for different values of y , show if $u_0 \in L^1$ then $\int u(x, y) dx = \int u_0$ for all $y > 0$ and $|u(x, y)| \leq C_d y^{-d} \int |u_0|$, derive asymptotics for suitable u_0 as $y \rightarrow \infty$ analogous to (35), and/or prove that if $u_0 \in H^s$ for some s , then $x \mapsto u(x, y) \in H^{s'}$ for every s' and $u(x, y)$ is C^∞ in all variables for $y > 0$.

To connect this result back to the wave equation, we substitute $u_0(x) = \delta(x)$, and combine (59) and (64) to obtain

$$\frac{1}{(2\pi)^d} \int e^{ix\xi} e^{-y|\xi|} d\xi = \frac{C_d y}{(|x|^2 + y^2)^{\frac{d+1}{2}}}. \quad (65)$$

We derived (65) for $y > 0$, but by analytic continuation the same equation also holds for $\operatorname{Re} y > 0$, provided we understand $(|x|^2 + y^2)^{\frac{d+1}{2}}$ as $(\sqrt{|x|^2 + y^2})^{d+1}$ with $\operatorname{Re} \sqrt{|x|^2 + y^2} > 0$, just as in the discussion immediately following (42).

Substituting $y = ic(t - i\varepsilon)$ and $y = -ic(t + i\varepsilon)$ into (65) and taking $\varepsilon \rightarrow 0^+$ gives

$$\frac{1}{(2\pi)^d} \int e^{ix\xi} e^{\pm ict|\xi|} d\xi = \frac{\mp C_d ict}{(|x|^2 - c^2(t \pm i0)^2)^{\frac{d+1}{2}}},$$

using here the same notation as in (25) where we computed $\hat{H}(\xi) = -i(\xi - i0)^{-1}$. Combining these equations and using $\frac{1}{2}(-iw + iw) = \operatorname{Im} w$ gives

$$\frac{1}{(2\pi)^d} \int e^{ix\xi} \cos(c|\xi|t) d\xi = \operatorname{Im} \frac{C_d ct}{(|x|^2 - c^2(t + i0)^2)^{\frac{d+1}{2}}}.$$

Similarly, provided $d \neq 1$, integrating (65) with respect to y we obtain

$$\frac{1}{(2\pi)^d} \int e^{ix\xi} \frac{e^{-y|\xi|}}{|\xi|} d\xi = \frac{C_d/(d-1)}{(|x|^2 + y^2)^{\frac{d-1}{2}}}, \quad \frac{1}{(2\pi)^d} \int e^{ix\xi} \frac{e^{\pm ict|\xi|}}{|\xi|} d\xi = \frac{C_d/(d-1)}{(|x|^2 - c^2(t \pm i0)^2)^{\frac{d-1}{2}}},$$

and

$$\frac{1}{(2\pi)^d} \int e^{ix\xi} \frac{\sin(c|\xi|t)}{c|\xi|} d\xi = \operatorname{Im} \frac{C_d/(d-1)}{c(|x|^2 - c^2(t + i0)^2)^{\frac{d-1}{2}}}.$$

Plugging into (58), in the form

$$u(x, t) = \frac{1}{(2\pi)^d} \int \int e^{i(x-x')\xi} \left(\cos(c|\xi|t) u_0(x') + \frac{\sin(c|\xi|t)}{c|\xi|} u_1(x') \right) d\xi dx',$$

gives

$$u(x, t) = C_d \int \left(\left(\operatorname{Im} \frac{ct}{(|x - x'|^2 - c^2(t + i0)^2)^{\frac{d+1}{2}}} \right) u_0(x') + \left(\operatorname{Im} \frac{1/(d-1)}{c(|x - x'|^2 - c^2(t + i0)^2)^{\frac{d-1}{2}}} \right) u_1(x') \right) dx', \quad (66)$$

as the solution to the wave equation in any dimension $d \geq 2$, where for $u_0, u_1 \in \mathcal{S}$ the integral is meant as a pairing in the sense of distributions. Recall that here $z^{(d\pm 1)/2}$ with $z = |x - x'|^2 - c^2(t - i\varepsilon)^2$ means $(\sqrt{z})^{d\pm 1}$, where we take the square root such that $\operatorname{Re} \sqrt{z} > 0$.

For $u_0, u_1 \in \mathcal{S}$, the solution u given by (66) is in \mathcal{S} for every t , as can be seen from the Fourier transform formula (54). As for the differential equations we have studied before, we can also allow $u_0, u_1 \in \mathcal{S}'$, and then (54) and (66) give the solution u in \mathcal{S}' for every t .

We rewrite (66) as

$$u(x, t) = \int \left(\partial_t R(x - x', t) u_0(x') + R(x - x', t) u_1(x') \right) dx',$$

where

$$R(x, t) = \text{Im} \frac{C_d/(d-1)}{c(|x|^2 - c^2(t + i0)^2)^{\frac{d-1}{2}}}.$$

The distribution $R(x, t)$ is called the *fundamental solution* to the wave equation, and is also known as the Riemann function.

EXERCISE 33. Derive the analog of (66) when $d = 1$ and simplify it. The u_0 term is the same, but the u_1 term is not. To find the u_1 term, evaluate $\int e^{ix\xi} \frac{\sin(c|\xi|t)}{c|\xi|} d\xi = \int e^{ix\xi} \frac{\sin(c\xi t)}{c\xi} d\xi$ by finding the Fourier transform of the characteristic function of an interval and using Fourier inversion (or by following the same steps as above). To simplify the u_0 term use (26). What is $R(x, t)$ in this case? Compare the result to d'Alembert's formula from Exercise 1.

The distribution $R(x, t)$ equals a smooth function almost everywhere: more precisely,

$$|x| > c|t| \quad \Longrightarrow \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{(|x|^2 - c^2(t + i\varepsilon)^2)^{\frac{d-1}{2}}} = \frac{1}{(|x|^2 - c^2t^2)^{\frac{d-1}{2}}}, \quad (67)$$

(in this region the limits $\varepsilon \rightarrow 0^+$ and $\varepsilon \rightarrow 0^-$ agree), and

$$|x| < c|t| \quad \Longrightarrow \quad \lim_{\varepsilon \rightarrow 0^+} \frac{1}{(|x|^2 - c^2(t + i\varepsilon)^2)^{\frac{d-1}{2}}} = \frac{(i \operatorname{sgn} t)^{d-1}}{(c^2t^2 - |x|^2)^{\frac{d-1}{2}}}, \quad (68)$$

where we used the fact that $\text{Im}(|x|^2 - c^2(t + i\varepsilon)^2) = -2c^2t\varepsilon$, so that $|x| < c|t|$ implies

$$\lim_{\varepsilon \rightarrow 0^+} \sqrt{|x|^2 - c^2(t + i\varepsilon)^2} = -i \operatorname{sgn} t \sqrt{c^2t^2 - |x|^2}.$$

From (67) we see that $R(x, t) = 0$ when $|x| > ct$, which implies that *waves propagate with speed at most c* . Thus, for given x and t , $u(x, t)$ depends on $u_0(x')$ and $u_1(x')$ only for x' such that $|x - x'| \leq c|t|$, i.e. for x' in $\overline{B(x, c|t|)}$, where

$$B(a, R) = \{x \in \mathbb{R}^d : |x - a| < R\}.$$

If $d \geq 3$ is odd, then from (68) we see that $R(x, t) = 0$ when $|x| < ct$, which implies that *waves propagate with speed precisely c* . Thus in that case $u(x, t)$ depends on $u_0(x')$ and $u_1(x')$ only for x' such that $|x - x'| = c|t|$. This is called the *strict* or *strong Huygens principle*.

On the other hand, for d even, (68) does not imply that $R(x, t) = 0$ when $|x| < ct$ but the fact that $R(x, t)$ is smooth there is still significant. For example, suppose u_0 and u_1 are zero outside of $B(0, R)$ for some $R > 0$. Then

$$u(x, t) = \int \left(\frac{C_d(-1)^{1+\frac{d}{2}} ct u_0(x')}{(c^2t^2 - |x - x'|^2)^{\frac{d+1}{2}}} + \frac{C_d(-1)^{\frac{d}{2}} u_1(x')/(d-1)}{c(c^2t^2 - |x - x'|^2)^{\frac{d-1}{2}}} \right) dx', \quad \text{when } |x| < ct - R, \quad (69)$$

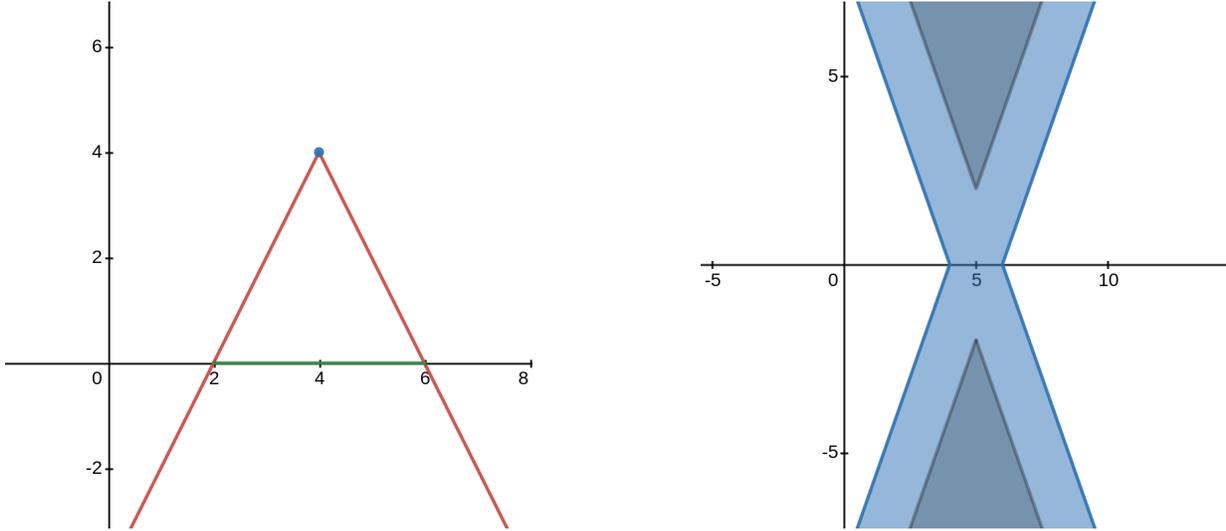


FIGURE 8. On the left: For given x and t , the solution $u(x, t)$ to the wave equation (53) depends on the initial conditions $u_0(x')$ and $u_1(x')$ only for x' in the closed ball of radius $c|t|$ centered at x . In the figure, the blue dot is (x, t) , and the value of u there is affected only by what is happening inside the red cone. The green segment is $\overline{B(x, c|t|)}$. The image is from <https://www.desmos.com/calculator/lzks4lrfgh>.

On the right: Suppose the initial conditions u_0 and u_1 are zero outside the intersection of the blue region with the horizontal axis. Then the solution $u(x, t)$ is zero in the white region for any d . If $d \geq 3$ odd, then the strong Huygens principle says that solution is zero in the darker gray region as well. If d is even, then we have only the weak Huygens principle and the solution is C^∞ in the darker gray region, and if $d = 1$ then the solution is constant in the darker gray region. The image is from <https://www.desmos.com/calculator/7pjpg7275tq>.

and $u(x, t)$ is C^∞ for such x and t even if u_0 and u_1 are not. Such a phenomenon is called a *weak Huygens principle*, because the singular (i.e. not C^∞) part of the solution propagates with speed precisely c , even if the whole solution does not.

EXERCISE 34. Use d'Alembert's formula from Exercise 1 or Exercise 33 to prove the following weak Huygens principle for $d = 1$. If u_0 and u_1 are zero outside of $B(0, R)$ for some $R > 0$, then $u(x, t)$ is constant on $x \in B(0, c|t| - R)$.

We can use (69) to compute long time asymptotics for solutions to the wave equation. Plugging in the binomial expansion

$$(c^2t^2 - |x - x'|^2)^{-s} = (ct)^{-2s}(1 - s(|x - x'|^2/c^2t^2) + \dots),$$

gives¹¹

$$u(x, t) = \begin{cases} 0, & \text{for } d \geq 3 \text{ odd,} \\ A_0t^{-d+1} + A_1(x)t^{-d} + \dots, & \text{for } d \text{ even,} \end{cases}, \quad \text{when } |x| < ct - R. \quad (70)$$

¹¹Thanks to Andy Kavouras and Pedro Morales for help with this calculation.

Here $A_0 = \frac{(-1)^{\frac{d}{2}} C_d}{c^d (d-1)} \int u_1$.

EXERCISE 35. Compute $A_1(x)$, and, if you can, give a more general formula for the terms in this series.

As a corollary we obtain *local energy decay*: if u_0 and u_1 are zero outside of some ball $B(0, R)$, then for any bounded open set $U \subset \mathbb{R}^d$, we have

$$\int_U \left(|\partial_t u(x, t)|^2 + c^2 |\nabla u(x, t)|^2 \right) dx \leq \begin{cases} 0, & \text{for } d \geq 3 \text{ odd,} \\ Ct^{-2d}, & \text{for } d \text{ even,} \end{cases} \quad (71)$$

uniformly for t large enough. Note that the integrand in (71) is the same as in the energy conservation identity (56), but the domain of integration is U in (71) where in (56) it was \mathbb{R}^d .

Investigating the *stability* of the expansion (70) and the local energy decay result (71) has been a major problem in mathematical scattering theory since the seminal work of Morawetz [Mor], Lax–Phillips [LaPh], and Vainberg [Vai]. See the book by Dyatlov and Zworski [DyZw] and the survey article of Zworski [Zwo] (especially, Equation (1.2), Theorem 5, and Theorem 12 of [Zwo]) for more recent results and references.

EXERCISE 36. Our calculation based on the solution formula (66) only derived (70) and (71) for $d \geq 2$. Derive the analogs in the case $d = 1$.

EXERCISE 37. Give a bound for the constant C in (71) in terms of u_0 , u_1 , c , and d .

The fundamental solution $R(x, t)$ simplifies nicely for smaller values of d .

The case $d = 1$ is in Exercise 33 above.

When $d = 2$, we have $C_2 = 1/\pi$, and, by (67) and (68) we have, for $t \neq 0$,

$$\int_{\mathbb{R}^2} R(x, t) \varphi(x) dx = \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R}^2} \operatorname{Im} \frac{C_2 \varphi(x) dx}{c(|x|^2 - c^2(t + i\varepsilon)^2)^{\frac{1}{2}}} = \int_{B(0, c|t|)} \frac{\varphi(x) dx}{c\pi(c^2 t^2 - |x|^2)^{\frac{1}{2}}},$$

where for the second equality we use the dominated convergence theorem; this is applicable because the function $x \mapsto \mathbf{1}_{B(0, c|t|)} / (c^2 t^2 - |x|^2)^{\frac{1}{2}}$ is integrable for each t .

Thus we obtain that the solution to (53) when $d = 2$ is

$$u(x, t) = \partial_t \left(\int_{B(0, c|t|)} \frac{u_0(x') dx'}{c\pi(c^2 t^2 - |x - x'|^2)^{\frac{1}{2}}} \right) + \int_{B(0, c|t|)} \frac{u_1(x') dx'}{c\pi(c^2 t^2 - |x - x'|^2)^{\frac{1}{2}}}. \quad (72)$$

EXERCISE 38. Prove that the function $\mathbb{R}^2 \rightarrow \mathbb{R}$ given by $x \mapsto \mathbf{1}_{B(0, c|t|)} / (c^2 t^2 - |x|^2)^{\frac{1}{2}}$ is integrable for each $t \neq 0$ by evaluating its integral. Sketch the graph of $x \mapsto R(x, t)$ for some different values of c and t , either by hand or with a computer, and, if you like, compare it with what you get when $d = 1$ and $d = 3$.

For the case $d = 3$, we recall the formula (26), which said that

$$\operatorname{Im} \frac{1}{x - i0} = \frac{1}{2i} \lim_{\varepsilon \rightarrow 0^+} \left(\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} \right) = \pi \delta(x).$$

We use this, and the partial fraction expansion $\frac{2b}{a^2-b^2} = \frac{1}{a-b} - \frac{1}{a+b}$, to simplify $R(x, t)$ by writing

$$\frac{2}{|x|^2 - (ct + i\varepsilon)^2} = \frac{1/(ct + i\varepsilon)}{|x| - (ct + i\varepsilon)} - \frac{1/(ct + i\varepsilon)}{|x| + (ct + i\varepsilon)}.$$

Taking the imaginary part of both sides, letting $\varepsilon \rightarrow 0^+$, and using $C_3 = 1/\pi^2$, gives

$$R(x, t) = \frac{\delta(|x| - ct) + \delta(|x| + ct)}{4c^2\pi t}, \quad \text{for } t \neq 0, d = 3.$$

For instance, if $t > 0$, then

$$\int_{\mathbb{R}^3} R(x, t)\varphi(x)dx = \frac{1}{4c^2\pi t} \int_{\mathbb{R}^3} \delta(|x| - ct)\varphi(x)dx = \frac{1}{4c^2\pi t} \int_{\partial B(0, ct)} \varphi(x)dS.$$

More generally we obtain that the solution to (53) when $d = 3$ is

$$u(x, t) = \partial_t \left(\frac{1}{4c^2\pi t} \int_{\partial B(x, c|t)} u_0(x')dS \right) + \frac{1}{4c^2\pi t} \int_{\partial B(x, c|t)} u_1(x')dS. \quad (73)$$

The formulas (72) and (73) are known as *Kirchoff's formulas*.

Further discussion and references. For more on the vibrating string, see Section 1.3 of [Str], for even more see Section 1.1. of [Wei] or Section 2.1 of [Tay].

The introduction to distribution theory above is an abbreviated version of [Str, Section 12.1] and [Lig]. Another good short introduction, including some functional analysis, is in Sections 3.3 and 3.4 of [Tay]. See [Kan] for a longer account including very many examples. A good more thorough treatment is [FrJo], and, for the insatiable, [Hör]. Above we stick to tempered distributions since all the distributions we need for our work are of this kind. Distributions which are not tempered are obtained by using as a space of test functions the smaller space $C_c^\infty(\mathbb{R}^d)$ consisting of $C^\infty(\mathbb{R}^d)$ which vanish outside of a compact set, and then a distribution is defined to be a \mathbb{C} -linear mapping $u: C_c^\infty(\mathbb{R}^d) \rightarrow \mathbb{C}$ which is the limit in the sense of distributions of a sequence of functions in $C_c^\infty(\mathbb{R}^d)$. Note that we defined distributions as functionals obtained as limits of test functions; it is more usual to define them as functional satisfying a different kind of continuity condition, and some proofs of equivalence between these conditions can be found in Theorems 1.3.2, 1.5.2, and 5.2.2 of [FrJo], or Theorems 2.1.4, 2.1.8, and 4.1.5 of [Hör].

Two centuries ago Fourier [Fou] observed that a general function on the line can be written as a superposition of sinusoids by the formula $f(x) = \frac{1}{2\pi} \int e^{ix\xi} \int e^{-iy\xi} f(y)dyd\xi$ (and, when 2π -periodic, $f(x) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} e^{ixk} \int_0^{2\pi} e^{-iyk} f(y)dy$). Since then many authors have explored what more precise scope and meaning can be given to this statement, a central problem being in what sense should the integrals be understood. Riemann [Rie] and Lebesgue [Leb] developed their respective theories of integration in order to study this problem. These notes use Schwartz's [Sch] very general and flexible notion of integration in the sense of distributions.

The heat equation was first derived and solved by Fourier [Fou]. In the modern study of the heat equation and its generalizations, maximum principles (not discussed above) play a central role [Eva, Sections 2.3 and 7.1].

The quantization relation $E = \hbar\omega$ in (36) comes from Planck's analysis of black body radiation in 1900 and Einstein's analysis of the photoelectric effect in 1905. In 1922 Compton used it and

the corresponding relation $\xi = \hbar k$ to explain the increase in X ray wavelength caused by scattering with substances with low atomic numbers. In 1924 de Broglie proposed that the same relations must hold for massive matter, specifically electrons, and in 1926 Schrödinger used these relations to derive his famous equation. The discussion of the Schrödinger equation in Section 5 is based on the one in [Mar, Section 1.2]; see [Ham], especially Sections 1.II, 1.VI, 4.II, and 4.III, for a more detailed discussion.

A derivation of the wave equation for $d = 2$ for the vibrating membrane can be found in Section 1.3 of [Str] or Section 2.1 of [Tay]. For a derivation for $d = 3$ for electromagnetic waves, see Section 9.2.1 of [Gri]. The presentation of the wave equation above mostly follows Section 3.5 of [Tay], but the diabolical tricks are from Theorem 1.14 of [StWe].

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