Introduction to wave equations

We begin our course by briefly surveying some important general properies of waves and wave equations, deferring detailed derivations and explanations for later. We begin with the simplest waves, which are either free or bound, and proceed to more general ones, which are built out of these. Throughout we pay attention to the particle–wave, or classical–quantum, correspondence.

1. Free waves. The basic waves are the sinusoids:

$$u(x,t) = \cos(\xi x - \tau t), \qquad u(x,t) = \sin(\xi x - \tau t), \qquad u(x,t) = e^{i(\xi x - \tau t)}, \tag{1}$$

where ξ and τ are real parameters, respectively the space and time *frequencies* of the waves. Here x and t are real variables, x for space and t for time. Factoring the argument as $\xi x - \tau t = \xi(x - \xi^{-1}\tau t)$ shows that the *velocity* of the wave is $v = \xi^{-1}\tau$.

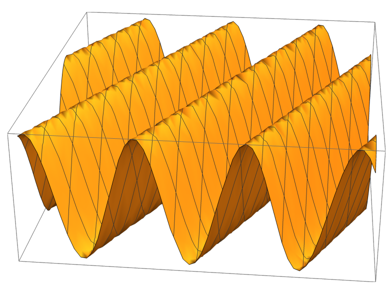

FIGURE 1. A sinusoidal wave. The period in x is $2\pi/|\xi|$ and the period in t is $2\pi/|\tau|$.

The basic wave equations are

$$u_{tt} - u_{xx} = 0, \qquad iu_t + u_{xx} = 0. \tag{2}$$

The first of (2) is so basic it is just called the *wave equation*. It describes vibrations of a taut string, the acceleration u_{tt} of the string being caused by the curvature u_{xx}. The second of (2) is called the *Schrödinger equation*. It describes the behavior of a quantum particle, such as an electron, in a one-dimensional channel. Waves solving these equations are called *free* because they propagate freely along the real line $x \in \mathbb{R}$.

Substituting the first of (1) into the first of (2) gives

$$-\xi^2 \cos(\xi x - \tau t) + \tau^2 \cos(\xi x - \tau t) = 0,$$

or

$$\tau = \pm\xi, \tag{3}$$

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and the velocity is $v = \pm 1$. The other forms from (1), plugged into the wave equation, also give (3). To get a corresponding result for the Schrödinger equation we use the last of (1), and plug into the second of (2), to find

$$\tau e^{i(\xi x - \tau t)} - \xi^2 e^{i(\xi x - \tau t)} = 0,$$

or

$$\tau = \xi^2, \tag{4}$$

and the velocity is $v = \xi$. Thus all sinusoid solutions to the wave equation travel at the same speed, while Schrödinger waves travel at speeds corresponding to their frequencies.

More general free waves can be analyzed as superpositions of these sinusoids, either discrete ones:

$$u(x,t) = a_1 e^{i\xi_1(x - v_1 t)} + a_2 e^{i\xi_N(x - v_N t)} + \cdots ,$$

or continuous ones:

$$u(x,t) = \int_{-\infty}^{\infty} a(\xi) e^{i\xi(x - v_\xi t)} d\xi.$$

In the case of the wave equation we can write

$$u(x,t) = \int_{-\infty}^{\infty} a(\xi) e^{i\xi(x-t)} d\xi + \int_{-\infty}^{\infty} b(\xi) e^{i\xi(x+t)} d\xi = f(x-t) + g(x+t). \tag{5}$$

In the case of the Schrödinger equation we can write

$$u(x,t) = \int_{-\infty}^{\infty} a(\xi) e^{i\xi(x - \xi t)} d\xi. \tag{6}$$

Proving completeness, i.e. the fact that with minor exceptions *all* solutions can be obtained in this way, requires deeper analysis and we will come back to this point.

In higher dimensions the wave and Schrödinger equations (2) become

$$u_{tt} - \Delta u = 0, \qquad i u_t + \Delta u = 0. \tag{7}$$

where $u = u(x,t)$, $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, $\Delta u = \sum_{j=1}^d u_{x_j x_j} = \nabla \cdot \nabla u$. We again use sinusoids of the form (1), but now with $x \in \mathbb{R}^d$, $\xi \in \mathbb{R}^d$, and $\xi x = \xi \cdot x = \sum_{j=1}^d \xi_j x_j$. The higher-dimensional wave equations describe vibrations of a membrane, as well as sound waves and electromagentic waves (light, radio, etc.). The higher dimensional Schrödinger equations describe free quantum particles. Then the frequency relationships (3) and (4) become

$$\tau = \pm |\xi|, \qquad \tau = |\xi|^2,$$

and the solution formulas (5) and (6) become

$$u(x,t) = \int_{\mathbb{R}^d} a(\xi) e^{i(\xi x - |\xi| t)} d\xi + \int_{\mathbb{R}^d} b(\xi) e^{i(\xi x + |\xi| t)} d\xi, \qquad u(x,t) = \int_{\mathbb{R}^d} a(\xi) e^{i\xi(x - \xi t)} d\xi.$$

Thus in each case the full solution wave u is a superposition of sinusoidal waves traveling along straight lines. This corresponds to the fact that free light rays and free classical particles move in straight lines.

It follows that if a solution u represents a localized wave, i.e. u is bounded and $|u| \to 0$ sufficiently quickly for fixed t as $|x| \to \infty$, then this solution u stabilizes for each x, i.e. $|u|$ tends to a constant for fixed x as $t \to \infty$. This can be seen directly in the simplest case of the one dimensional wave equation (5) when the component functions f and g are each localized; then the solution stabilizes to 0. For the more complicated cases we will see it later; it comes from the fact that waves having different frequencies travel with different velocities, a phenomenon which in general is called *dispersion*.

EXERCISE. Given $c \in \mathbb{R}$, construct f and g such that $u(x,t) = f(x-t) + g(x+t)$ is a localized wave which stabilizes to c.

Hint: Use the signum function, i.e. $\operatorname{sgn} x = x/|x|$ for $x \neq 0$ and $\operatorname{sgn} 0 = 0$.

2. Bound waves. The simplest bound waves are solutions to the same basic wave equations (2), but with x restricted to a bounded interval, say $(0, L)$ for some $L > 0$. At the endpoints $x = 0$ and $x = L$ we impose the boundary conditions. In the case of the string this means the string is held fixed there (as the string of a musical instrument) and in the case of a quantum particle that means it is prevented from leaving the interval by impenetrable barriers at the endpoints (the 'particle in a box').

With such boundary conditions, sinusoidal solutions of the form (1) no longer work directly, so we expand them using angle addition formulas and keep only the favorable terms, i.e. the ones with $\sin(\xi x)$, imposing

$$\xi = \lambda_n = n\pi/L, \tag{8}$$

with n an integer so as to satisfy the boundary conditions. That gives

$$u(x,t) = \sin(\xi x)\sin(\tau t), \qquad u(x,t) = \sin(\xi x)\cos(\tau t), \qquad u(x,t) = \sin(\xi x)e^{-i\tau t}.$$

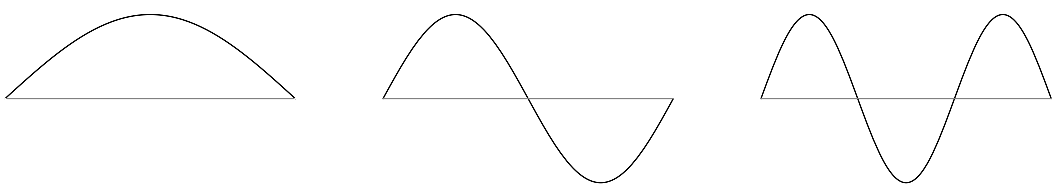

FIGURE 2. Graphs of the bound states $\sin(\xi x) = \sin(\lambda_n x)$ for $n = 1,\ 2,\ 3$.

Note that the density of frequencies, i.e. the spacing of the values of λ_n, is given by π/L, and in particular it grows with the size of the region of confinement.

FIGURE 3. The frequencies $\lambda_n = n\pi/L$ with $L = 1$ and with $L = 2$.

These new sinusoidal solutions play a role analogous to that played by the old ones (1) for free waves. Plugging in (3) and (4), and taking linear combinations, gives

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)(a_n \sin(\lambda_n t) + b_n \cos(\lambda_n t)), \qquad u(x,t) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x) e^{-i\lambda_n^2 t}, \tag{9}$$

for the wave and Schrödinger equations respectively. Note that the solutions in (9) are the analogs of the solutions we found before for the corresponding free equations (5) and (6), this time with integrals replaced by sums because only a discrete family of values of the frequency ξ can occur.

Unlike free waves, bound waves do not stabilize. In this simplest case, *all* waves are periodic in time with period $2L$. This corresponds to the fact that free light rays, or free classical particles, bouncing back and forth in the interval $(0, L)$, have period $2L$.

Bound waves also arise in the presence of less rigid confinement. Here matters become more complicated and our explanations more sketchy. The fundamental example is the quadratic confining term:

$$-u_{tt} = -u_{xx} + x^2 u, \qquad iu_t = -u_{xx} + x^2 u,$$

where again x ranges over $\mathbb{R}$. This is called a *harmonic oscillator*. For a vibrating string this corresponds to an elastic restorative force which grows in strength the farther one is from the origin, and for a quantum particle this corresponds to an attractive force at the origin whose strength is proportional to distance from the origin. Now solutions have the same form as in (9), namely

$$u(x,t) = \sum_{n=1}^{\infty} \psi_n(x)(a_n \sin(\lambda_n t) + b_n \cos(\lambda_n t)), \qquad u(x,t) = \sum_{n=1}^{\infty} a_n \psi_n(x) e^{-i\lambda_n^2 t}, \tag{10}$$

but this time with the more complicated formulas $\lambda_n(x) = \sqrt{2n-1}$ and $\psi_n(x) = H_{n-1}(x)e^{-x^2/2}$, where $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2, \ldots$ are the *Hermite* polynomials.

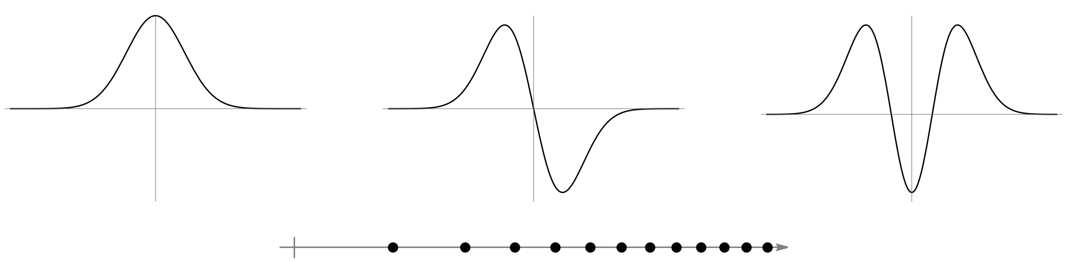

FIGURE 4. Graphs of the bound states $\psi_n(x)$ for $n = 1,\ 2,\ 3$, and below them the frequencies $\lambda_n = \sqrt{2n-1}$.

Note that the density of frequencies now grows with n. This corresponds to the fact that the region of confinement of a classical harmonic oscillator grows with energy. To depict the relationship more precisely we work with the *energy*, given by the square of the frequency λ_n^2; this is the natural quantity because it has the same units as x^2.

3. More general waves. More general waves combine features of the above two kinds. An example is given by Gamow's model for alpha decay in a radioactive nucleus. This describes the process by which a radioactive nucleus, such as uranium, decays by emitting an alpha particle, that is to say a helium nucleus consisting of two protons and two neutrons. For this model we

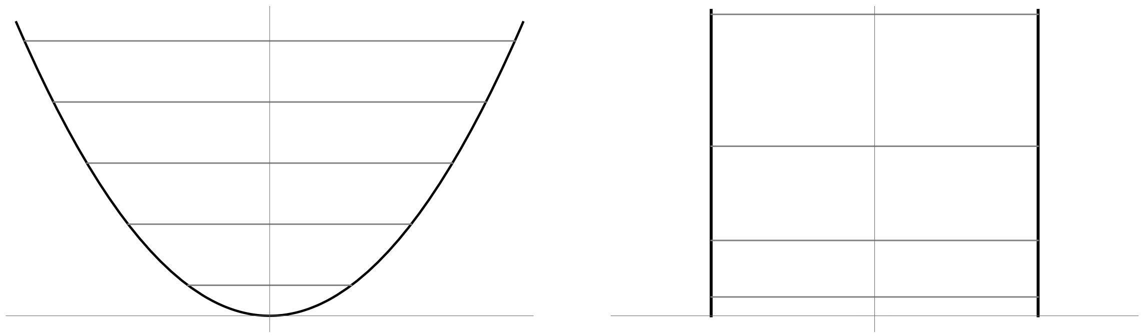

FIGURE 5. The energy levels $\lambda_n^2 = 2n - 1$ for the harmonic oscillator and $\lambda_n^2 = (n\pi/L)^2$ with $L = 2$.

replace x^2 by a potential energy function

$$V(x) = \begin{cases} -V_0, & x < L, \\ c/x, & x > L. \end{cases}$$

Here c/x corresponds to the electric repulsion from the other protons, and at short range $x < R$ it is overwhelmed by the nuclear binding force, which corresponds to $-V_0$ and is what holds the nucleus together. Now x, which corresponds to the distance from the center of the nucleus, is restricted to $(0, \infty)$, and we impose the boundary condition $u(0, t) = 0$. A similar picture arises in the study of diatomic molecules such as hydrogen H_2 or oxygen O_2. In that case x corresponds to the distance between the two nuclei.

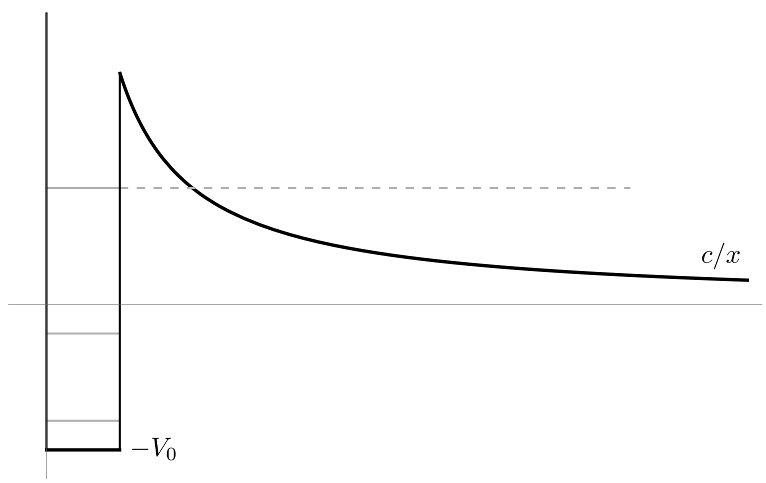

FIGURE 6. The potential energy function $V(x)$ and the associated energy levels approximately given by $\lambda_n^2 \approx (n\pi/L)^2$. The first two, with $n = 1$ and $n = 2$, correspond to bound states, and the third one $n = 3$ to a resonant state.

Then the solutions of the Schrödinger equation

$$iu_t = -u_{xx} + V(x)u$$

have the form

$$u(x,t) = \sum_{n=1}^{2} a_n \psi_n(x) e^{-i\lambda_n^2 t} + \int_0^\infty a(\xi)\psi(\xi,x)e^{-i\xi^2 t},$$

and the solutions to the corresponding wave equation $iu_t = -u_{xx} + V(x)u$ have an analogous form. Here the $\psi_n(x)$ and λ_n are analogous to the ones appearing in the bound waves solutions (9) and (10); they correspond to the bound portion of the solution. The $a(\xi)$ here corresponds to the one in the free wave solution (6), and the $\psi(\xi,x)$ generalizes the $e^{i\xi x}$ appearing there; they correspond to the *scattering* portion of the solution.

The scattering portion stabilizes, just as it did in the free case. However it now contains certain *resonant* components which are longer-lived. The most important correspond approximately to bound states of the corresponding bound problem, in this case the particle restricted to $(0, L)$, but instead of being purely oscillatory they decay slowly. A major part of our course will be to understand bound, scattering, and resonant components of waves in a general way.

Further discussion and references. We are guided by Hume's principle of surmounting the difficulties of abstract reasonings by *avoiding all unnecessary detail* (see the end of Chapter I of [Hum]), though this is always a challenge. A gentle introduction to the wave equation can be found in Strauss' textbook [Str]: the free wave equation solution (5) is derived in Section 2.1, and the bound wave equation solution (9) in Section 4.1. A gentle introduction to quantum mechanics and Schrödinger's equation can be found in Griffiths' textbook [Gri]; the solutions to the Schrödinger equation given in (6), (9), and (10) are derived in Sections 2.1–2.4 there. For alpha decay, see Figure 8.5 of the first edition of [Gri] or Example 9.2 of the third edition, which has further references. For the diatomic molecule model, see [Mor] and section 2.1 of [Moi].

References

[Gri] David J. Griffiths, *Introduction to Quantum Mechanics*, 1995. Third edition, joint with Darrell F. Schroeter, 2018.

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