Free Waves

Scattering Theory I

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Introduction

In this course we will be studying the behavior of waves as they scatter off of a disturbance, and we will introduce resonances as a central tool for this purpose. Resonances gives rates of oscillation and decay of waves.

These slides will be posted at `https://www.math.purdue.edu/~kdatchev/SMS`

Some additional recommended references, in ascending order of length and complexity, are

- Hintz's five-lecture course `http://math.mit.edu/~phintz/snap19/index.html` has some cool videos as well as lecture notes and exercises.
- Dyatlov's semester course `https://math.mit.edu/~dyatlov/18.156/` has more comprehensive lecture notes and more exercises.
- Dyatlov and Zworski's book *Mathematical Theory of Scattering Resonances* is a much broader and deeper introduction to the subject.

As we go, I welcome questions and comments in the form of interruptions/chat messages/emails/etc.

The free wave equation

Let $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$, and let w be the forward solution to the free wave equation

$$(\partial_t^2 - \Delta) w(t,x) = f(t,x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n, \tag{$*$}$$

where $\Delta = \sum_{j=1}^n \partial_{x_j}^2$. The *forward* solution is the solution satisfying $w(t,x) = 0$ when $t \ll 0$.

Thus w gives the waves resulting from the forcing term f. Physical examples include vibrations of a membrane resulting from an external force, and components of electromagnetic waves resulting from charges and currents.

We call $(*)$ the *free* wave equation to distinguish it from the *perturbed* wave equation that we will consider later. The latter is given by

$$(\partial_t^2 + H) w(t,x) = f(t,x), \tag{$**$}$$

where H is an operator which equals (or approximately equals) $-\Delta$ outside of a compact subset of $\mathbb{R}^n$.

Question. What is the relationship between the solutions to $(*)$ and $(**)$?

Solving the free wave equation I

To get a formula for the forward solution to

$$(\partial_t^2 - \Delta)w(t,x) = f(t,x), \qquad t \in \mathbb{R},\ x \in \mathbb{R}^n, \qquad f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n),$$

we first take the Fourier transform with respect to x:

$$\partial_t^2 \hat{w}(t,\xi) + |\xi|^2 \hat{w}(t,\xi) = \hat{f}(t,\xi),$$

and then solve the resulting ODE[1] to get

$$\hat{w}(t,\xi) = \int_{-\infty}^{t} \frac{\sin(t-s)|\xi|}{|\xi|} \hat{f}(s,\xi)\, ds, \qquad \text{or} \qquad w(t,x) = \int_{\mathbb{R}\times\mathbb{R}^n} U(t-s, x-y) f(s,y)\, ds\, dy,$$

where

$$U(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} d\xi, \quad \text{for} \quad t \geq 0,$$

and $U(t,x) = 0$ for $t < 0$ (in the sense of distributions).

Solving the free wave equation II

To compute

$$U(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} d\xi, \quad \text{for} \quad t \geq 0,$$

at least up to a constant factor, we use the more basic (but tricky[2]) Fourier transform identity

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-y|\xi|} d\xi = \frac{Cy}{(y^2+|x|^2)^{\frac{n+1}{2}}}, \quad \text{for} \quad y > 0.$$

(Here and below, C is a real constant which changes from line to line). Integrating both sides with respect to y gives, when $n \geq 2$,[3]

$$\int e^{ix\cdot\xi} \frac{e^{-y|\xi|}}{|\xi|} d\xi = \frac{C}{(y^2+|x|^2)^{\frac{n-1}{2}}} \text{ for } y > 0, \text{ and so } \int e^{ix\cdot\xi} \frac{e^{it|\xi|}}{|\xi|} d\xi = \frac{C}{(|x|^2-t^2)^{\frac{n-1}{2}}} \text{ for } \operatorname{Im} t > 0.$$

Taking the limit as t approaches the real axis gives the following distributional boundary values:

$$\int e^{ix\cdot\xi} \frac{e^{it|\xi|}}{|\xi|} d\xi = \lim_{\varepsilon\to 0^+} \frac{C}{(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}}}, \qquad U(t,x) = \operatorname{Im} \lim_{\varepsilon\to 0^+} \frac{C}{(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}}}, \quad \text{for} \quad t \geq 0.$$

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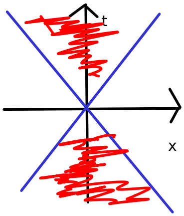
Distributional boundary values

The distributional boundary value

$$\int e^{ix\cdot\xi}\frac{e^{it|\xi|}}{|\xi|}d\xi = \lim_{\varepsilon\to 0^+}\frac{C}{(|x|^2-(t+i\varepsilon)^2)^{\frac{n-1}{2}}} = \frac{C}{(|x|^2-(t+i0)^2)^{\frac{n-1}{2}}}, \qquad \text{for} \quad t\geq 0,$$

is almost everywhere smooth. More precisely, if $|x|>t$, then

$$\frac{1}{(|x|^2-(t+i0)^2)^{\frac{n-1}{2}}} = \frac{1}{(|x|^2-t^2)^{\frac{n-1}{2}}}, \qquad (*)$$



and if $t>|x|$, then

$$\frac{1}{(|x|^2-(t+i0)^2)^{\frac{n-1}{2}}} = \frac{(-i)^{n-1}}{(t^2-|x|^2)^{\frac{n-1}{2}}}. \qquad (**)$$

Notice that $(*)$ is real for all n, and $(**)$ is real for odd n. Recall that $U(t,x)=\operatorname{Im}\frac{C}{(|x|^2-(t+i0)^2)^{\frac{n-1}{2}}}$.

This difference between n odd and n even will recur and be important.

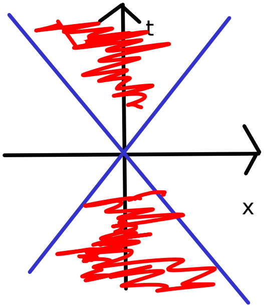
The forward fundamental solution of the free wave equation

Thus the forward solution to $(\partial_t^2 - \Delta)w = f$ is $w(t,x) = \int_{\mathbb{R}\times\mathbb{R}^n} U(t-s, x-y) f(s,y)\, ds\, dy$, where[4]

$$U(t,x) = \operatorname{Im} \frac{C}{(|x|^2 - (t+i0)^2)^{\frac{n-1}{2}}}, \qquad t \geq 0.$$

When $n = 2$ this is

$$U(t,x) = \begin{cases} \dfrac{1}{2\pi\sqrt{t^2 - |x|^2}}, & |x| < t, \\ 0, & \text{otherwise}, \end{cases}$$



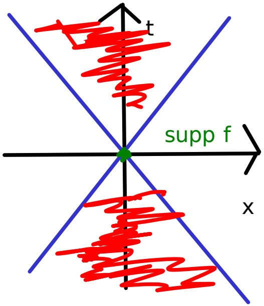
and when $n = 3$ this is

$$U(t,x) = \frac{1}{4\pi t}\delta(|x| - t).$$

Huygens' principle for the free wave equation

Thus the forward solution to $(\partial_t^2 - \Delta)w = f$ is $w(t,x) = \int_{\mathbb{R}\times\mathbb{R}^n} U(t-s, x-y) f(s,y)\, ds\, dy$, where[5]

$$U(t,x) = \operatorname{Im} \frac{C}{(|x|^2 - (t+i0)^2)^{\frac{n-1}{2}}}, \qquad t \geq 0.$$



We can now allow f to be a compactly supported distribution.

If $n \geq 3$ is odd, then U vanishes away from $|x| = t$. Hence

$$\operatorname{supp} w \subset \{(t,x) \text{ such that } |x-y| = t-s \text{ for some } (s,y) \in \operatorname{supp} f\}.$$

This is the strong Huygens principle.

If $n \geq 2$ is even, then U vanishes away from $|x| \leq t$ and is smooth away from $|x| = t$. Hence

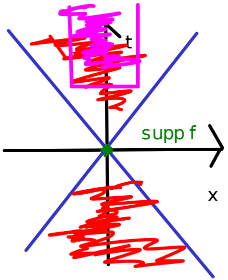
$$\operatorname{supp} w \subset \{(t,x) \text{ such that } |x-y| \leq t-s \text{ for some } (s,y) \in \operatorname{supp} f\},$$
$$\operatorname{sing\,supp} w \subset \{(t,x) \text{ such that } |x-y| = t-s \text{ for some } (s,y) \in \operatorname{supp} f\}.$$

This is the weak Huygens principle.[6]

Decay of free waves

The forward solution to $(\partial_t^2 - \Delta)w = f$ is $w(t,x) = \int_{\mathbb{R}\times\mathbb{R}^n} U(t-s, x-y) f(s,y)\, ds\, dy$, where

$$U(t,x) = \operatorname{Im} \frac{C}{(|x|^2 - (t+i0)^2)^{\frac{n-1}{2}}}, \qquad t \geq 0.$$



Let f be a compactly supported distribution and $B \subset \mathbb{R}^n$ be a ball. Then, if T is large enough depending on B and the support of f:

- If $n \geq 3$ is odd,[7] then, by the strong Huygens principle,

$$w(t,x) = 0, \qquad \text{for all } x \in B,\ t \geq T.$$

  Thus all waves die away completely in any compact set.

- If $n \geq 2$ is even, then, by the weak Huygens principle, w is C^∞ for all $x \in B$, $t \geq T$. Moreover,[8]

$$|w(t,x)| \leq C_f t^{-n+1}, \qquad \text{for all } x \in B,\ t \geq T.$$

  Thus all waves decay polynomially in any compact set.

Question. What are the correct generalizations of these results for perturbed waves? How do the decay rates depend on the perturbation?

References

The presentation above follows Chapter 3, Section 5, of Taylor's *Partial Differential Equations*, except for Exercise 2 which follows Theorem 1.14 of Chapter 1 of Stein and Weiss's *Introduction to Fourier Analysis on Euclidean Spaces*.

Exercises.

The exercises marked with a * are more central to the course. (They are not the more difficult ones.)

1. Show that if $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$, then the solution to

$$\partial_t^2 \hat{w}(t,\xi) + |\xi|^2 \hat{w}(t,\xi) = \hat{f}(t,\xi),$$

which obeys $\hat{w}(t,\xi) = 0$ when $t \ll 0$, is given by

$$\hat{w}(t,\xi) = \int_{-\infty}^{t} \frac{\sin(t-s)|\xi|}{|\xi|} \hat{f}(s,\xi) ds.$$

Hint: Use the method of variation of parameters, or the Laplace transform or Fourier transform. If you're not used to using variation of parameters, this is a good occasion to practice it; it is generally useful in one dimensional scattering.

2. Prove that

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} e^{-y|\xi|} d\xi = \frac{Cy}{(y^2+|x|^2)^{\frac{n+1}{2}}}, \quad \text{for} \quad y > 0,$$

Hint: Use a change of variables to reduce to the case $y = 1$, and then plug in the following:

$$e^{-|\xi|} = C \int_{-\infty}^{\infty} \frac{e^{i|\xi|s}}{1+s^2} ds, \quad \text{and} \quad \frac{1}{1+s^2} = \int_0^{\infty} e^{-u} e^{-us^2} du,$$

(to check the first use contour deformation), and then switch the order of integration, so that you get a Gaussian Fourier transform and then another Gaussian Fourier transform. If you're curious, you can write the constant in terms of the Gamma function, but we won't need this.

3. * Compute

$$U(t,x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \frac{\sin t|\xi|}{|\xi|} d\xi, \qquad \text{for } t \geq 0,$$

when $n = 1$.

Hint: Use the Fourier transform of the characteristic function of an interval and the Fourier inversion formula.

4.

1. Given φ and $\psi \in C_c^\infty(\mathbb{R}^n)$, find a distribution f on $\mathbb{R} \times \mathbb{R}^n$ such that the forward solution w to

$$(\partial_t^2 - \Delta_x)w(t,x) = f(t,x),$$

equals the solution to the Cauchy problem

$$(\partial_t^2 - \Delta_x)w_1(t,x) = 0, \qquad w_1(0,x) = \varphi(x), \qquad \partial_t w_1(0,x) = \psi(x),$$

when $t > 0$.

Hint: Use $\delta(t)$ and $\delta'(t)$. You can solve the Cauchy problem by taking Fourier transform with respect to x as in Exercise 1. Check uniqueness using the fact that the energy $\int_{\mathbb{R}^n} (\partial_t w_1)^2 + |\nabla_x w_1|^2 dx$ is conserved.

2. When $n = 2$, show by direct calculation that

$$U(t,x) = \operatorname{Im} \frac{C}{(|x|^2 - (t+i0)^2)^{\frac{n-1}{2}}} = \frac{C}{\sqrt{t^2 - |x|^2}}, \quad \text{for} \quad t > |x|.$$

and $U(t,x) = 0$ otherwise. In particular, combining with the result of Exercise 3, U is a locally integrable function when $n \leq 1$.

3. Prove by direct calculation the Plemelj jump formula

$$(x - i0)^{-1} - (x + i0)^{-1} = Ci\delta(x),$$

and use it to show that

$$U(t,x) = \operatorname{Im} \frac{C}{(|x|^2 - (t+i0)^2)^{\frac{n-1}{2}}} = \frac{C}{t}\delta(|x|-t),$$

when $n = 3$.

5. Given φ and $\psi \in C_c^\infty(\mathbb{R}^n)$, find a distribution f on $\mathbb{R} \times \mathbb{R}^n$ such that the forward solution w to

$$(\partial_t^2 - \Delta_x) w(t,x) = f(t,x),$$

equals the solution to the Cauchy problem

$$(\partial_t^2 - \Delta_x) w_1(t,x) = 0, \qquad w_1(0,x) = \varphi(x), \qquad \partial_t w_1(0,x) = \psi(x),$$

when $t > 0$.

Hint: Use $\delta(t)$ and $\delta'(t)$. You can solve the Cauchy problem by taking Fourier transform with respect to x as in Exercise 1. Check uniqueness using the fact that the energy $\int_{\mathbb{R}^n} (\partial_t w_1)^2 + |\nabla_x w_1|^2 dx$ is conserved.

6. * Show that the weak Huygens principle holds when $n = 1$ using the calculation of $U(t,x)$ from Exercise 3.

7. * Let f be a compactly supported distribution in $\mathbb{R} \times \mathbb{R}$, let w be the forward solution to $\partial_t^2 w - \partial_x^2 w = f$, and let $I \subset \mathbb{R}$ be a bounded interval. Use the calculation of $U(t, x)$ from Exercise 3 to show that if T is large enough depending on I and the support of f, then

$$w(t, x) - C \int_{\mathbb{R} \times \mathbb{R}} f = 0, \qquad \text{for all } x \in I, \; t \geq T.$$

Thus $C \int_{\mathbb{R} \times \mathbb{R}} f$ is the steady state solution. Later we will interpret this term as a projection onto a resonance at 0.

8. Let

$$w(t, x) = \int_{\mathbb{R} \times \mathbb{R}^n} U(t - s, x - y) f(s, y) \, ds \, dy,$$

where $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ and $n \geq 2$ is even. Use the fact that

$$U(t, x) = \frac{C}{(t^2 - |x|^2)^{\frac{n-1}{2}}}, \qquad \text{when} \qquad t > |x|,$$

to show that for any ball $B \subset \mathbb{R}^n$ there is a T such that

$$|w(t, x)| \leq C_f t^{-n+1}, \qquad \text{for all } x \in B, \; t \geq T.$$

Find an example showing that the power t^{-n+1} is optimal.