Perturbed Waves and Resonances

Scattering Theory II

Kiril Datchev (Purdue University)

Microlocal Analysis Summer School

Séminaire de Mathématiques Supérieures

July 5 – July 16 2021

Please email any comments, questions, and corrections to kdatchev@purdue.edu

Fourier–Laplace transform of the perturbed wave equation

Let H be a differential operator on $\mathbb{R}^n$, a perturbation of $-\Delta$, such as $-\Delta + V$ with $V \in C_c^\infty(\mathbb{R}^n)$, and let $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$. We wish to find and study the forward solution to

$$(\partial_t^2 + H)w(t,x) = f(t,x). \qquad (*)$$

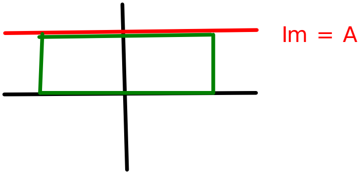
We will use a Fourier–Laplace transform with respect to t, and, for a suitable $A \in \mathbb{R}$, define

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds.$$

Paley--Wiener

Note that $\lambda \mapsto \tilde{f}(\lambda,x)$ is entire and $|\partial^\alpha \tilde{f}(\lambda,x)| \le C_{\alpha,N} e^{-T_0 \operatorname{Im}\lambda}|\lambda|^{-N}$ for any α, N, $\operatorname{Im}\lambda \ge 0$. Hence, by differentiating under the integral and using a contour deformation, w solves $(*)$:

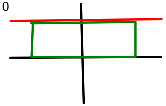
$$(\partial_t^2 + H)w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}\tilde{f}(\lambda,x)d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-it\lambda}\tilde{f}(\lambda,x)d\lambda = f(t,x).$$



Vector ODE I

We thus see that, if the resolvent $(H-\lambda^2)^{-1}$ is nice enough that we can differentiate under the integral, then

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds,$$



solves

$$(\partial_t^2 + H)w(t,x) = f(t,x).$$

To analyze the solution further, we do further contour deformations, and use further mapping properties and estimates of the resolvent.

This is difficult, so as a warm-up, consider the easier vector ODE problem where H is an $m\times m$ matrix and $f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m)$. We wish to find $w \in C^\infty(\mathbb{R}; \mathbb{C}^m)$ such that

$$(\partial_t^2 + H)w(t) = f(t),$$

and such that $w(t) = 0$ for $t \le T_0$.

Vector ODE II

Let H be an $m \times m$ matrix and $f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m)$. Then, for A sufficiently large, the vector ODE

$$(\partial_t^2 + H)w(t) = f(t),$$

has a unique forward solution[1] given by

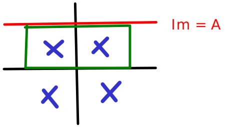
$$w(t) = \frac{1}{2\pi} \int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H - \lambda^2)^{-1} \tilde{f}(\lambda) d\lambda, \qquad \tilde{f}(\lambda) = \int_{T_0}^{T_1} e^{is\lambda} f(s) ds.$$

We will first check this by differentiating under the integral sign, second show that $w(t) = 0$ for $t \le 0$, and third compute asymptotics as $t \to \infty$.

1. To justify differentiating under the integral sign, use the fact that $\tilde{f}$ is entire,

$$|\tilde{f}(\lambda)| \le C_f e^{-T_0 \operatorname{Im} \lambda} |\lambda|^{-2}, \qquad \text{when} \qquad \operatorname{Im} \lambda > 0$$

by integration by parts, and, when $|\lambda|$ is sufficiently large, we have the estimate



$$\|(H - \lambda^2)^{-1}\| \le 2|\lambda|^{-2},$$

thanks to the geometric series

$$(H - \lambda^2)^{-1} = -\lambda^{-2}(I - \lambda^{-2} H)^{-1} = -\lambda^{-2} \sum_{j=0}^{\infty} \lambda^{-2j} H^j.$$

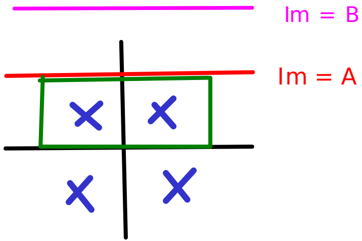
Vector ODE III

Let H be an $m \times m$ matrix and $f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m)$. Then, for A sufficiently large

$$w(t) = \frac{1}{2\pi} \int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)d\lambda, \ \tilde{f}(\lambda) = \int_{T_0}^{T_1} e^{is\lambda} f(s)ds \quad \text{solves} \quad (\partial_t^2 + H)w(t) = f(t).$$

2. Next, if $B > A$, then, using $|\tilde{f}(\lambda)| \le C_f e^{-T_0 \operatorname{Im}\lambda}|\lambda|^{-2}$ and $\|(H-\lambda^2)^{-1}\| \le 2|\lambda|^{-2}$,

$$w(t) = \frac{1}{2\pi} \int_{-\infty+iB}^{\infty+iB} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)d\lambda,$$



by shifting the contour up. Thus

$$|w(t)| \le \frac{e^{tB}}{2\pi} \int_{-\infty}^{\infty} |(H-(\mu+iB)^2)^{-1}\tilde{f}(\mu+iB)|d\mu$$
$$\le C_f e^{(t-T_0)B},$$

which tends to 0 as $B \to \infty$, for all $t < T_0$.

Hence $w(t) = 0$ for $t \le T_0$.

Vector ODE IV

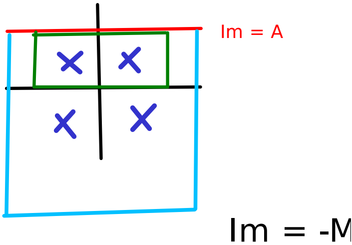
Let H be an $m \times m$ matrix and $f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m)$. Then, for A sufficiently large

$$w(t) = \frac{1}{2\pi} \int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)d\lambda, \ \tilde{f}(\lambda) = \int_{T_0}^{T_1} e^{is\lambda} f(s)ds \quad \text{solves} \quad (\partial_t^2 + H)w(t) = f(t).$$

3. To study asymptotics as $t \to \infty$, we repeat the same argument, but instead shift the contour down. We use $|\tilde{f}(\lambda)| \le C_f e^{-T_1 \operatorname{Im}\lambda}|\lambda|^{-2}$ when $\operatorname{Im}\lambda < 0$, take $M \gg 0$, and obtain

$$w(t) = -\sum_{\lambda_j} iR_{\lambda_j}(t) + E(t),$$

$\|(H-\lambda^2)^{-1}\| \le \frac{2}{|\lambda|}$ for $|\lambda|$ large



where $R_{\lambda_j}(t)$ is the residue of $e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)$ at the pole λ_j, and

$$|E(t)| = \frac{1}{2\pi}\left| \int_{-\infty-iM}^{\infty-iM} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)d\lambda \right| \le C_f e^{-M(t-T_1)}.$$

Taking $M \to \infty$ shows that $E(t) = 0$ when $t \ge T_1$.

Vector ODE V

Let H be an $m \times m$ matrix and $f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m)$. Then, for A sufficiently large

$$w(t) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)d\lambda, \ \tilde{f}(\lambda) = \int_{T_0}^{T_1} e^{is\lambda} f(s)ds \quad \text{solves} \quad (\partial_t^2 + H)w(t) = f(t).$$

For $t \geq T_1$, we have

$$w(t) = -\sum_{\lambda_j} iR_{\lambda_j}(t),$$

where R_{λ_j} is the residue of $e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda)$ at the pole λ_j. If λ_j is a pole of order K, then

$$R_{\lambda_j}(t) = \sum_{k=1}^{K} t^{k-1} e^{-it\lambda_j} w_{j,k}.$$

Note that poles occur at square roots of eigenvalues of H. One can show[2] that the $w_{j,k}$ are (generalized) eigenvectors of H with eigenvalue λ_j^2.

When $m = 1$, the problem becomes particularly simple, and we have two poles with $k = 1$, unless $H = 0$.[3]

w'' = 0

Perturbed waves

After this warm-up, we turn to an actual problem in scattering theory. Given $V \in C_c^\infty(\mathbb{R}^n)$ and $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$, we will solve

$$(\partial_t^2 - \Delta + V)w(t,x) = f(t,x),$$

by writing, for A sufficiently large,

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds.$$

We will construct the perturbed resolvent $(-\Delta + V - \lambda^2)^{-1}$ by using the free resolvent $(-\Delta - \lambda^2)^{-1}$ as a *parametrix*, or approximate solution, writing

$$(-\Delta + V - \lambda^2)(-\Delta - \lambda^2)^{-1} = I + V(-\Delta - \lambda^2)^{-1}.$$

Then we will solve away the remainder by showing that $(-\Delta - \lambda^2)^{-1}$ is small in a suitable sense when $\operatorname{Im}\lambda$ is large.

using a geometric series

Estimates on the free resolvent.

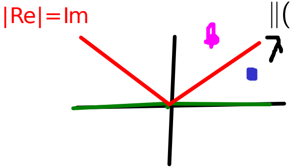
For any real r, the Sobolev space H^r is given by

$$H^r = \{f : (-\Delta + 1)^{r/2} f \in L^2\}, \qquad \|f\|_{H^r} = \|(-\Delta+1)^{r/2} f\|_{L^2},$$

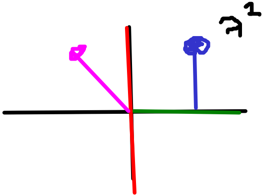
with functions of $-\Delta$, including the free resolvent $(-\Delta - \lambda^2)^{-1}$, defined as Fourier multipliers:

$$\varphi(-\Delta) f = \mathcal{F}^{-1}\varphi(|\xi|^2)\mathcal{F} f.$$

Consequently, for all real r, and $\operatorname{Im}\lambda > 0$,



$$\begin{aligned}\|(-\Delta - \lambda^2)^{-1}\|_{H^r \to H^r} &= \|\mathcal{F}^{-1}(|\xi|^2 - \lambda^2)^{-1}\mathcal{F}\|_{L^2\to L^2} \\ &= \sup_{t \geq 0} |t - \lambda^2|^{-1} \\ &\leq \frac{1}{|\lambda| \operatorname{Im}\lambda},\end{aligned}$$



where for the inequality we used the fact that

$$\lambda^2 = (\operatorname{Re}\lambda)^2 - (\operatorname{Im}\lambda)^2 + 2i \operatorname{Re}\lambda \operatorname{Im}\lambda,$$

and hence

$$\operatorname{dist}(\lambda^2, [0,\infty)) = \begin{cases} 2\operatorname{Im}\lambda |\operatorname{Re}\lambda|, & (\operatorname{Re}\lambda)^2 \geq (\operatorname{Im}\lambda)^2, \\ |\lambda|^2, & (\operatorname{Re}\lambda)^2 \leq (\operatorname{Im}\lambda)^2. \end{cases}$$

Construction of the perturbed resolvent $(-\Delta + V - \lambda^2)^{-1}$ when $\operatorname{Im}\lambda \gg 0$.

Using

$$\|(-\Delta - \lambda^2)^{-1}\|_{H^r \to H^r} = \sup_{t \geq 0} |t - \lambda^2|^{-1} \leq \frac{1}{|\lambda| \operatorname{Im}\lambda}, \qquad \text{when } \operatorname{Im}\lambda > 0,$$

when $|\lambda| \operatorname{Im}\lambda$ is large enough[4], we can use a geometric series $(I - X)^{-1} = \sum_{k=0}^{\infty} X^k$ to invert

$$(-\Delta + V - \lambda^2)(-\Delta - \lambda^2)^{-1} = I + V(-\Delta - \lambda^2)^{-1}.$$

and obtain

$$(-\Delta + V - \lambda^2)^{-1} = (-\Delta - \lambda^2)^{-1}(I + V(-\Delta - \lambda^2)^{-1})^{-1}, \qquad \|(-\Delta + V - \lambda^2)^{-1}\|_{H^r \to H^r} \leq \frac{2}{|\lambda| \operatorname{Im}\lambda}.$$

This justifies defining, for A sufficiently large,

$$w(t,x) = \frac{1}{2\pi} \int_{-\infty + iA}^{\infty + iA} e^{-it\lambda} (-\Delta + V - \lambda^2)^{-1} \tilde{f}(\lambda, x) d\lambda, \qquad \tilde{f}(\lambda, x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x) ds,$$

when $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$, because $|\tilde{f}(\lambda)| \leq C_f e^{-T_0 \operatorname{Im}\lambda} / |\lambda|$ when $\operatorname{Im}\lambda > 0$.

Solution of the perturbed wave equation I

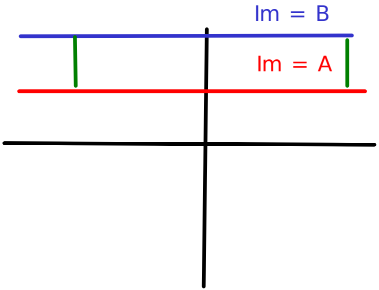
Theorem. Let $V \in C_c^\infty(\mathbb{R}^n)$, $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$. For A large enough, the function

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds,$$

solves $(\partial_t^2 - \Delta + V)w = f$ and $w(t,x) = 0$ for $t \leq T_0$.

Proof. We first show independence of the choice of such A by proving that if $B > A$, then

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iB}^{\infty+iB} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda.$$



This follows by deformation of contour together with

$$\lim_{L\to\pm\infty}\int_{L+iA}^{L+iB} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda = 0,$$

where we used $\|(-\Delta + V - \lambda^2)^{-1}\|_{H^r\to H^r} \leq \frac{2}{|\lambda|\,\mathrm{Im}\,\lambda}$
and $\|\tilde{f}(\lambda,\cdot)\|_{H^r} \leq C_f e^{-T_0 B}/|\lambda|$.

Solution of the perturbed wave equation II

We have defined, independently of A large enough,

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds.$$

Estimating as before, i.e. $\|(-\Delta + V - \lambda^2)^{-1}\|_{H^r\to H^r} \le \frac{2}{|\lambda|\operatorname{Im}\lambda}$ and $\|\tilde{f}(\lambda,\cdot)\|_{H^r} \le C_f e^{-T_0A}/|\lambda|^3$, justifies differentiation under the integral sign to show that

$$(\partial_t^2 - \Delta + V)w = f.$$

It remains to show that $w(t,x) = 0$ for all $t < T_0$. For that we let $A \to \infty$ in the estimate

$$|w(t,x)| = \frac{1}{2\pi}e^{tA}\left|\int_{-\infty}^{\infty} e^{-it\mu}(-\Delta + V - (\mu + iA)^2)^{-1}\tilde{f}(\mu + iA, x)d\mu\right| \le C_f e^{(t-T_0)A}.$$

□

To analyze $w(t,x)$ when t is large, we must deform the contour downward, rather than upward.

This means analyzing the perturbed resolvent $(-\Delta + V - \lambda^2)^{-1}$ for a larger set of values of λ.

The perturbed resolvent $(-\Delta + V - \lambda^2)^{-1}$ in the upper half plane

We used a geometric series to define, for $\operatorname{Im}\lambda$ large enough,

$$(-\Delta + V - \lambda^2)^{-1} = (-\Delta - \lambda^2)^{-1}(I + V(-\Delta - \lambda^2)^{-1})^{-1}. \qquad (*)$$

The inclusion from compactly supported H^{r+2} functions into H^r is compact, so $\lambda \mapsto V(-\Delta - \lambda^2)^{-1}$ is a holomorphic family of compact operators.

Recall that

- K being compact means that for any $\varepsilon > 0$ there is a finite rank F such that $\|K - F\| < \varepsilon$,
- $H^{r+\varepsilon} \subset H^r$ is compact on a torus because $(1-\Delta)^{-\varepsilon/2}$ is (use partial Fourier sums to find F).
- $\lambda \mapsto F(\lambda)$ is a holomorphic family of operators means $\lambda \mapsto \langle F(\lambda)u, v\rangle$ is holomorphic for all u, v.

Thus $(*)$ extends meromorphically to the upper half plane $\operatorname{Im}\lambda > 0$ by the

Analytic Fredholm Theorem. Let $\mathcal{H}$ be a Hilbert space, let $\Omega \subset \mathbb{C}$ be a connected open set, and let $\lambda \mapsto K(\lambda)$ be a holomorphic family of compact operators on $\mathcal{H}$ for $\lambda \in \Omega$. If $(I - K(\lambda_0))^{-1}$ exists for some $\lambda_0 \in \Omega$, then $\lambda \mapsto (I - K(\lambda))^{-1}$ is a meromorphic family of (Fredholm) operators for $\lambda \in \Omega$.

Moreover, poles of this meromorphic continuation occur at precisely those λ which are square roots of eigenvalues of $(-\Delta + V - \lambda^2)$ in $\mathbb{C} \setminus [0, \infty)$. Indeed, the proof below shows that $I + V(-\Delta - \lambda^2)^{-1}$ is invertible if and only if it is injective, and the same follows for $-\Delta + V - \lambda^2$ by inverting $(*)$.[5]

Analytic Fredholm Theorem

Theorem. If $\lambda \mapsto K(\lambda)$ is holomorphic for $\lambda \in \Omega$, $K(\lambda)$ compact, $\Omega \subset \mathbb{C}$ connected and open, and if $(I - K(\lambda_0))^{-1}$ exists for some $\lambda_0 \in \Omega$, then $\lambda \mapsto (I - K(\lambda))^{-1}$ is meromorphic for $\lambda \in \Omega$.

Proof. Let $D \subset \Omega$ be a disk such that $\|K(\lambda) - K(\lambda')\| < 1/2$ for any $\lambda,\ \lambda' \in D$. Suppose there is $\lambda' \in D$ such that $I - K(\lambda')$ is invertible, and pick a finite rank F such that $\|K(\lambda') - F\| < 1/2$, so that $\lambda \mapsto (I - K(\lambda) + F)^{-1}$ is holomorphic for $\lambda \in D$. Then

geo.ser. $$(I - K(\lambda))(I - K(\lambda) + F)^{-1} = I - F(I - K(\lambda) + F)^{-1} := I - G(\lambda).$$

Thus $I - K(\lambda)$ is invertible if and only if $I - G(\lambda)$ is, i.e. if and only if $u - G(\lambda)u = f$ has a unique solution. After substituting $u = f + v$, solving $u - G(\lambda)u = f$ is equivalent to solving

$$v = G(\lambda)(f + v). \qquad (*)$$

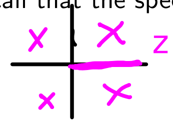
Any solution v must be in the range of F, i.e. of the form $v = \sum_{j=1}^{N} c_j(\lambda)e_j$, where $\{e_1, \ldots, e_N\}$ is an orthonormal basis of the range of F. Plugging this into $(*)$ and pairing both sides with e_k gives

$$c_k(\lambda) = \langle G(\lambda)(f + \Sigma_{j=1}^{N} c_j(\lambda)e_j), e_k\rangle = \langle G(\lambda)f, e_k\rangle + \Sigma_{j=1}^{N}\langle G(\lambda)e_j, e_k\rangle c_j(\lambda).$$

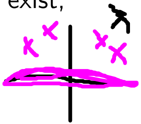
By linear algebra, this system of equations defines a meromorphic function $\lambda \mapsto c_k(\lambda)$ for $\lambda \in D$. This proves the result for $\lambda \in D$. By connectedness,[6] it follows for $\lambda \in \Omega$.

Interlude on the spectrum of $-\Delta + V$

Recall that the spectrum of $-\Delta$ is $[0,\infty)$. We showed that, wherever all inverses exist,



$$-\Delta + V - z = (I + V(-\Delta - z)^{-1})(-\Delta - z),$$
$$(-\Delta + V - z)^{-1} = (-\Delta - z)^{-1}(I + V(-\Delta - z)^{-1})^{-1}.$$



By the Analytic Fredholm Theorem, this shows that the spectrum of $-\Delta + V$ is discrete in $\mathbb{C} \setminus [0,\infty)$. Similarly, switching the roles of $-\Delta$ and $-\Delta + V$,

$$-\Delta - z = (I - V(-\Delta + V - z)^{-1})(-\Delta + V - z)$$
$$(-\Delta - z)^{-1} = (-\Delta + V - z)^{-1}(I - V(-\Delta + V - z)^{-1})^{-1}.$$

This shows that the spectrum of $-\Delta + V$ contains $[0,\infty)$. If u is an eigenfunction of $-\Delta + V$ with eigenvalue z, then

$$0 = \langle(-\Delta + V - z)u, u\rangle = \|\nabla u\|^2 + \langle Vu, u\rangle - z\|u\|^2.$$

Thus

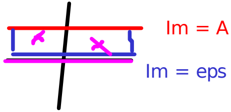
$$\operatorname{Im} z = \frac{\operatorname{Im}\langle Vu, u\rangle}{\|u\|^2} \in [\min \operatorname{Im} V, \max \operatorname{Im} V], \qquad \operatorname{Re} z = \frac{\|\nabla u\|^2 + \langle Vu, u\rangle}{\|u\|^2} \geq \min \operatorname{Re} V.$$

If V is nonnegative then no discrete spectrum

Long time perturbed wave asymptotics I

Hence $(-\Delta + V - \lambda^2)^{-1} \colon H^r \to H^r$ is meromorphic for $\operatorname{Im}\lambda > 0$. When $|\lambda| \operatorname{Im}\lambda$ is large enough, it is also holomorphic and obeys

$$\|(-\Delta + V - \lambda^2)^{-1}\|_{H^r \to H^r} \le \frac{2}{|\lambda| \operatorname{Im}\lambda}.$$



Parallel to the proof that $w(t,x) = 0$ when $t < T_0$, we deform the contour downward in the formula

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)\,d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)\,ds,$$

to a line $\operatorname{Im}\lambda = \varepsilon$ chosen to avoid any resolvent poles (i.e. eigenvalues), to obtain

$$w(t,x) = -\sum_{\{\lambda_j \colon \operatorname{Im}\lambda_j > \varepsilon\}} iR_{\lambda_j}(t,x) + E(t,x),$$

where $R_{\lambda_j}(t,x)$ is the residue of $e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)$ at the pole λ_j, and

$$|E(t,x)| = \frac{1}{2\pi}e^{t\varepsilon}\left|\int_{-\infty}^{\infty} e^{-it\mu}(-\Delta + V - (\mu + iA)^2)^{-1}\tilde{f}(\mu + iA, x)\,d\mu\right| \le C_f e^{t\varepsilon}.$$

Note that the remainder E, though it grows exponentially, does so more slowly than the eigenvalue terms. To get a better remainder we will go into the lower half plane.

Meromorphic continuation of the cutoff perturbed resolvent $\chi(-\Delta + V - \lambda^2)^{-1}\chi$.

To go into the lower half plane, we will take advantage of the compact supports of f and V. We take $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi V = V$ and $\chi f = f$, and we will study

$$\chi(x)w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}\chi(x)(-\Delta+V-\lambda^2)^{-1}\chi\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds,$$

To analyze the cutoff reoslvent, we multiply the identity

$$(-\Delta+V-\lambda^2)^{-1} = (-\Delta-\lambda^2)^{-1}(I+V(-\Delta-\lambda^2)^{-1})^{-1},$$

by χ on both sides, to obtain

$$\begin{aligned}\chi(-\Delta+V-\lambda^2)^{-1}\chi &= \chi(-\Delta-\lambda^2)^{-1}(I+V(-\Delta-\lambda^2)^{-1})^{-1}\chi \\ &= \chi(-\Delta-\lambda^2)^{-1}\chi(I+V(-\Delta-\lambda^2)^{-1}\chi)^{-1},\end{aligned}$$

$\chi(I+Y\chi+Y\chi Y\chi+\cdots)$

where we verify the last equality for $\operatorname{Im}\lambda \gg 0$ using a geometric series:

$$(I-Y)^{-1}\chi = \chi + Y\chi + Y^2\chi + \cdots = \chi(I-Y\chi)^{-1}, \quad \text{where} \quad Y = -V(-\Delta-\lambda^2)^{-1} \text{ and } \chi Y = Y,$$

and then extend it by meromorphic continuation to $\operatorname{Im}\lambda > 0$.

This identity and the analytic Fredholm theorem show that meromorphic continuation of $\chi(-\Delta+V-\lambda^2)^{-1}\chi$ follows from holomorphic continuation of $\chi(-\Delta-\lambda^2)^{-1}\chi$, and this holomorphic continuation is what we show next.

The free resolvent in terms of the wave propagator

To analyze the free resolvent, we use a Fourier–Laplace transform to relate it to the free wave propagator $U(t, x)$ that we analyzed before. By direct calculation,[7]

$$(|\xi|^2 - \lambda^2)^{-1} = \int_0^\infty e^{it\lambda} \frac{\sin t|\xi|}{|\xi|} dt, \qquad \text{for} \qquad \operatorname{Im}\lambda > 0.$$

Hence, by the self-adjoint functional calculus, (or by a Fourier transform in ξ),

$K(\lambda, x) = \int_0^\infty e^{it\lambda} u(t,x) dt$

$$(-\Delta - \lambda^2)^{-1} = \int_0^\infty e^{it\lambda} \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} dt, \qquad \text{for} \qquad \operatorname{Im}\lambda > 0,$$

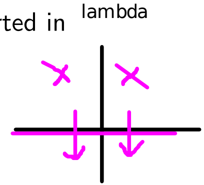
and the formula simplifies nicely[8] when $n = 1$ or $n = 3$. To be able to continue into the lower half plane, we multiply on both sides by $\chi \in C_c^\infty(\mathbb{R}^n)$ to obtain

$$\chi(-\Delta - \lambda^2)^{-1}\chi = \int_0^\infty e^{it\lambda} \chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \chi dt, \qquad \text{for} \qquad \operatorname{Im}\lambda > 0.$$

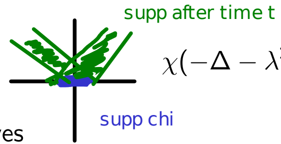
Continuing the cutoff free resolvent into the lower half plane

When $n \geq 3$ odd, the strong Huygens principle shows that $\chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \chi$ is supported in

$$\{x \in \operatorname{supp} \chi \text{ such that } |x - y| = t \text{ for some } y \in \operatorname{supp} \chi\}.$$



Plugging this into the formula



$$\chi(-\Delta - \lambda^2)^{-1}\chi = \int_0^\infty e^{it\lambda} \chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \chi dt, \qquad \text{for} \qquad \operatorname{Im} \lambda > 0,$$

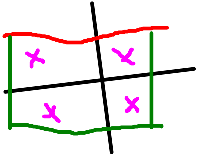
gives

$$\chi(-\Delta - \lambda^2)^{-1}\chi = \int_0^T e^{it\lambda} \chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \chi dt, \qquad \text{for} \qquad \operatorname{Im} \lambda > 0,$$

if T is large enough (depending on $\operatorname{supp} \chi$), and this last expression extends to an entire function of λ.

When n is even, we get a $\int_T^\infty$ term which we must also analyze. Using the fact that for T large the integral kernel of $\chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} \chi$ is in $C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, and a deformation of contour, allows us to holomorphically continue this term, although the result is not entire in λ.[9]

Continuing the cutoff perturbed resolvent into the lower half plane



When $n \geq 3$ odd, we showed that for any $\chi \in C_c^\infty(\mathbb{R}^n)$ there is $T > 0$ such that

$$\chi(-\Delta - \lambda^2)^{-1}\chi = \int_0^T e^{it\lambda}\chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\chi dt,$$

with the formula initially defined for $\operatorname{Im}\lambda > 0$ and then extended holomorphically to $\lambda \in \mathbb{C}$. Since $\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}} : L^2 \to H^1$, we see that $\chi(-\Delta - \lambda^2)^{-1}\chi \colon L^2 \to L^2$ is compact.

Thus the Analytic Fredholm Theorem shows that, if $\chi V = V$, then

$$\chi(-\Delta + V - \lambda^2)^{-1}\chi = \chi(-\Delta - \lambda^2)^{-1}\chi(I + V\chi(-\Delta - \lambda^2)^{-1}\chi)^{-1}$$

continues meromorphically to $\lambda \in \mathbb{C}$, with poles called *resonances*.[10]

To be able to apply contour deformation in our study of waves, we also need an estimate on the norm of the cutoff resolvent. We automatically have

$$\|\chi(-\Delta + V - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq 2\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2},$$

when $\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2}$ is small enough, so we will estimate $\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2}$.

Estimates on the continuation of the free resolvent

For $n \geq 3$ odd,[11] integrating by parts once in the formula

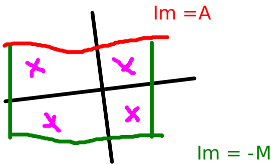
$$\chi(-\Delta - \lambda^2)^{-1}\chi = \int_0^T e^{it\lambda}\chi \frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\chi dt, \qquad \text{for} \qquad \lambda \in \mathbb{C},$$

gives

$$\chi(-\Delta - \lambda^2)^{-1}\chi = \frac{e^{it\lambda}}{i\lambda}\chi\frac{\sin t\sqrt{-\Delta}}{\sqrt{-\Delta}}\chi\Big|_{t=0}^{t=T} - \int_0^T \frac{e^{it\lambda}}{i\lambda}\chi \cos t\sqrt{-\Delta}\chi dt,$$

and hence

$$\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq C_\chi \frac{e^{-T\,\mathrm{Im}\,\lambda}}{|\lambda|}.$$



Thus, for any $M > 0$, we have

$$\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq \frac{C_{\chi,M}}{|\lambda|}, \qquad \text{when} \qquad \mathrm{Im}\,\lambda \geq -M.$$

Since $\|\chi(-\Delta + V - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq 2\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2}$ when $\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2}$ is small enough, that implies

$$\|\chi(-\Delta - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq \frac{C_{\chi,M}}{|\lambda|}, \qquad \text{when} \qquad \mathrm{Im}\,\lambda \geq -M \text{ and } |\lambda| \text{ is large enough.}$$

Long time perturbed wave asymptotics II

$f \in C_c^\infty([T_0,T_1]\times\mathbb{R}^n)$

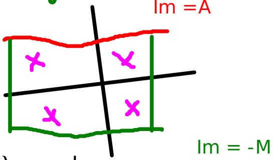
Thus, for any $M>0$, $\chi(-\Delta+V-\lambda^2)^{-1}\chi\colon L^2\to L^2$ is meromorphic for $\operatorname{Im}\lambda > -M$. When $|\lambda|$ is large enough, it is also holomorphic and obeys

$|\tilde f| \le C|\lambda|^{-N} e^{-T_1 \operatorname{Im}\lambda}$

$$\|\chi(-\Delta+V-\lambda^2)^{-1}\chi\|_{L^2\to L^2}\le C/|\lambda|.$$

We take χ such that $\chi f = f$, and deform the contour downward in the formula

$$\chi w(t,x)=\frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}\chi(-\Delta+V-\lambda^2)^{-1}\chi\tilde f(\lambda,x)d\lambda,\qquad \tilde f(\lambda,x)=\int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds,$$



to the line $\operatorname{Im}\lambda=-M$, to obtain

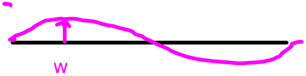
$$\chi w(t,x) = -\sum_{\{\lambda_j:\ \operatorname{Im}\lambda_j>-M\}} iR_{\lambda_j}(t,x)+E(t,x),$$

where $R_{\lambda_j}(t,x)$ is the residue of $e^{-it\lambda}\chi(-\Delta+V-\lambda^2)^{-1}\chi\tilde f(\lambda,x)$ at the pole λ_j, and

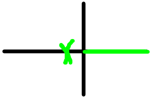
$$\|E(t,x)\|_{L^2}=\frac{1}{2\pi}e^{-Mt}\Big\|\int_{-\infty}^{\infty}e^{-it\mu}\chi(-\Delta+V-(\mu-iM)^2)^{-1}\chi\tilde f(\mu-iM,x)d\mu\Big\|_{L^2}\le C_f e^{-M(t-T_1)}.$$

As we saw before, poles with $\operatorname{Im}\lambda>0$ correspond to eigenvalues of $-\Delta+V$. Poles with $\operatorname{Im}\lambda\le 0$ are called *resonances*.

Interpretation



Let V be real valued, and let w measure displacement from equilibrium of a string or membrane, and write our wave equation in the form



$$\partial_t^2 w = f + \Delta w - Vw.$$

Then, recalling Newton's law $F = ma$, and using units such that the mass density is one, then

- f is the outside driving force (independent of w),
- Δw is the restoring force due to tension,
- and Vw is a force, proportional to displacement, which pushes towards equilibrium when $V > 0$ and away when $V < 0$.

Correspondingly, growing modes, i.e. resolvent poles in the open upper half plane, can occur only when $\min V < 0$. One can also show that if $\min V = 0$, then there are no poles in the *closed* upper half plane; see Section 5 of Hintz's lecture notes.

Some examples can be found here: `https://www.cs.cornell.edu/~bindel/cims/resonant1d/`, see in particular Figures 1, 2, and 10.

Outgoing and incoming solutions

Let $R(\lambda) = (-\Delta + V - \lambda^2)^{-1}$ and $\operatorname{Im} A \gg 0$. The forward solution to $(\partial_t^2 - \Delta + V)w = f$ is

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda} R(\lambda)\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds.$$

The resolvent $R(\lambda)$ for $\operatorname{Im}\lambda > 0$, and the meromorphic continuation of $\chi R(\lambda)\chi$, are called *outgoing*.

We can similarly obtain the backward solution by writing

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty-iA}^{\infty-iA} e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds.$$

The resolvent $R(\lambda)$ for $\operatorname{Im}\lambda < 0$, and the meromorphic continuation of $\chi R(\lambda)\chi$, are called *incoming*.

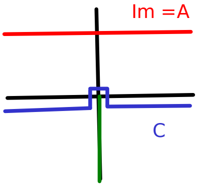
Sometimes people use different conventions for the square root/Fourier transform, interchanging the upper and lower half plane; the significant thing is that outgoing goes with forward, and incoming goes with backward. The formulas and results for the forward and backward solutions are all basically the same due to the time symmetry of the wave equation.

Even dimensions

In even dimensions, the meromorphic continuation is not to $\mathbb{C}$, but to the logarithmic Riemann surface. This means more complicated contour deformation arguments are needed, including estimates near the origin.

We already saw from the free case that, even in the absence of eigenvalues and resonances, in general decay can be no better than

$$|w(t,x)| \leq C_f t^{-n+1};$$



such a rate comes from the singularity of the resolvent at $\lambda = 0$ in the integral

$$w(t,x) = \frac{1}{2\pi} \int_C e^{-it\lambda}(-\Delta + V - \lambda^2)^{-1} \tilde{f}(\lambda, x) d\lambda,$$

where C is a contour in $\mathbb{C} \setminus \{\lambda \in \mathbb{C} \colon \operatorname{Re} \lambda = 0 \text{ and } \operatorname{Im} \lambda \leq 0\}$, chosen to avoid the upper half plane as much as possible. But this singularity can be complicated, especially in dimension two; see Jensen and Nenciu's 2001 Reviews in Mathematical Physics paper, and Vasy's 2021 Pure and Applied Analysis paper.

References

The presentation above follows Sections 3.1 and 3.2 of Dyatlov and Zworski's *Mathematical Theory of Scattering Resonances*, Hintz's lecture notes `http://math.mit.edu/~phintz/snap19/index.html`, and Lectures 7 and 8 of Dyatlov's lecture notes `http://math.mit.edu/~dyatlov/18.156/`. The proof of the analytic Fredholm theorem follows Theorem VI.14 of Volume 1 of Reed and Simon's *Methods of Mathematical Physics*.

Exercises.

The exercises marked with a * are more central to the course. (They are not the more difficult ones.)

1. Show that

$$(\partial_t^2 + H)w(t) = f(t), \qquad f \in C_c^\infty([T_0, T_1]; \mathbb{C}^m),$$

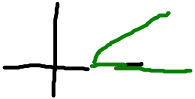
has a unique solution which vanishes for $t < T_0$ by showing that the difference of any two solutions has a Fourier–Laplace transform which is everywhere zero.

2. Let H be an $m \times m$ matrix. Show that $(H - \lambda^2)$ is invertible if and only if λ^2 is not an eigenvalue of H. Use Cramer's rule to show that, in a neighborhood of any λ_j such that λ_j^2 is an eigenvalue of H, we have a Laurent expansion

$$(H - \lambda^2)^{-1} = \sum_{k=-K}^{\infty} A_k (\lambda^2 - \lambda_j^2)^k,$$

for some $K \geq 1$ and some matrices A_k. Multiply this equation on the left by $(H - \lambda^2) = (H - \lambda_j^2) +$

$(\lambda_j^2 - \lambda^2)$ and match like powers of $(\lambda^2 - \lambda_j^2)$ to show that



$$(H - \lambda_j^2)A_k = \begin{cases} 0, & k = -K, \\ I + A_{k-1}, & k = 0, \\ A_{k-1}, & \text{otherwise.} \end{cases}$$

Conclude that A_{-K} maps into the eigenspace of H at λ_j^2, and if $K > 1$, then A_k maps into the generalized eigenspace of H at λ_j^2 when $-K < k < 0$.

3. Let $H \in \mathbb{C}$ and $f \in C_c^\infty([T_0, T_1])$. Show that if A is sufficiently large, then

$$w(t) = \frac{1}{2\pi} \int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H - \lambda^2)^{-1}\tilde{f}(\lambda)d\lambda,$$

solves

$$\partial_t^2 w + Hw = f, \qquad w(t) = 0 \text{ for } t \leq T_0.$$

Evaluate $w(t)$ for $t \geq T_1$ by shifting the contour down. Distinguish the cases $H \neq 0$ and $H = 0$. Compare your the result to that of Excercise 1 from Part 1.

4.* Let $V \in C_c^\infty(\mathbb{R}^n)$.