Stronger perturbations

Scattering Theory III

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Variable wavespeeds

We now consider the wave equation

$$(\partial_t^2 - c(x)^2\Delta)w(t,x) = f(t,x),$$

where $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$ and $c \in C^\infty(\mathbb{R}^n)$ is everywhere positive and $c - 1$ compactly supported, and where we seek a solution w obeying $w = 0$ when $t \le T_0$.

Here c represents a wavespeed which can vary spatially, corresponding to variations in the medium through which the wave propagates.

This can be done as before by writing, for A sufficiently large,

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(-c^2\Delta - \lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds,$$

and we can study asymptotics as $t \to \infty$ by shifting the contour down, but there are new challenges.

Variable wavespeeds

Previously we had $V \in C_c^\infty(\mathbb{R}^n)$, and we wrote

$$(-\Delta + V - \lambda^2)(-\Delta - \lambda^2)^{-1} = I + V(-\Delta - \lambda^2)^{-1}.$$

We then used the facts that $\|V(-\Delta - \lambda^2)^{-1}\|_{L^2 \to L^2}$ is small when $\operatorname{Im} \lambda$ is large, and that $V(-\Delta - \lambda^2)^{-1}$ is a compact operator.

Now with $c \in C^\infty(\mathbb{R}^n)$ everywhere positive and $c - 1$ compactly supported, if we write as before

$$(-c^2\Delta - \lambda^2)(-\Delta - \lambda^2)^{-1} = I + (1 - c^2)\Delta(-\Delta - \lambda^2)^{-1},$$

the trouble begins with the fact that $(1 - c^2)\Delta(-\Delta - \lambda^2)^{-1}$ does not necessarily have small norm anywhere and is not a compact operator; the new perturbation $(1 - c^2)\Delta$, though still compactly supported, is much bigger than the old one V.

To handle this, we will use a more elaborate parametrix construction, based on Sjöstrand–Zworski black box scattering theory, applicable to also to more general situations where one operator is a compactly supported perturbation of another.

A more general operator

$D = -i\partial$

Let

$$H = \sum_{j,k=1}^{n} a_{jk} D_j D_k + \sum_{\ell=1}^{n} b_\ell D_\ell + V,$$

where all the b_ℓ, V are in $C_c^\infty(\mathbb{R}^n)$, and the a_{jk} are C^∞, real valued, and equal the identity matrix outside of a compact set. We assume moreover that H is elliptic in the sense that there is $C > 0$ such that

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \geq |\xi|^2/C, \qquad \text{for all } (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Examples to keep in mind:

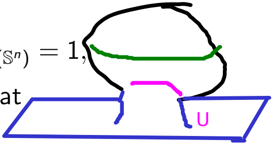
- The variable wavespeed problem $H = -c^2\Delta$, $c \in C^\infty(\mathbb{R}^n; (0,\infty))$, $c - 1$ compactly supported.
- The potential perturbation problem $H = -\Delta + V$, $V \in C_c^\infty(\mathbb{R}^n)$, that we looked at in Part II.
- Laplacians with respect to Riemannian metrics $H = -G^{-1}\sum_{j,k=1}^n \partial_j G g^{jk}\partial_k$, $G = \sqrt{\det g_{jk}}$, g^{jk} is the inverse matrix to g_{jk}, where g_{jk} is a Riemannian metric equal to the Euclidean metric outside of a compact set. That is to say, the g_{jk} are C^∞, positive definite, and equal the identity matrix outside of a compact set.

(Don't confuse H with the Sobolev space H^r; we will never raise the operator H to any power.)

An example with large resolvent $\varphi_m \sim (x+iy)^m$

We can make an example with large resolvent based on spherical harmonics small near a point. Let φ_m be a sequence of eigenfunctions of the sphere,

$$-\Delta_{\mathbb{S}^n}\varphi_m = \lambda_m^2\varphi_m, \qquad \text{with} \qquad \lambda_m \to \infty, \qquad \|\varphi_m\|_{L^2(\mathbb{S}^n)} = 1,$$



and such that there is a nonempty open $U \subset \mathbb{S}^n$ and a constant C such that

$$\|\varphi_m\|_{L^2(U)} \le e^{-\lambda_m/C}, \qquad \text{for every } m.$$

Such a construction can be based on eigenfunctions concentrating on a great circle.[1] Then take $\chi \in C^\infty(\mathbb{S}^n)$ which is 1 on $\mathbb{S}^n \setminus U$ and with $\operatorname{supp}\chi \neq \mathbb{S}^n$, and put $u_m = \chi\varphi_m$. Then

$$\|u_m\|_{L^2(\mathbb{S}^n)} \ge 1 - e^{-\lambda_m/C}, \qquad \|(-\Delta_{\mathbb{S}^n} - \lambda_m^2)u_m\|_{L^2(\mathbb{S}^n)} \le \|[-\Delta_{\mathbb{S}^n}, \chi]\varphi_n\|_{L^2(\mathbb{S}^n)} \le e^{-\lambda_m/C}.$$

Let g_{jk} be a metric on $\mathbb{R}^n$ which is Euclidean outside of a compact set and such that there is an open set $U' \subset \mathbb{R}^n$ isometric to a neighborhood of $\operatorname{supp} u_m$. Define the Laplacian

$$H = -G^{-1}\sum_{j,k=1}^{n} \partial_j G g^{jk} \partial_k, \qquad G = \sqrt{\det g_{jk}}, \qquad g^{jk} \text{ is the inverse matrix to } g_{jk}.$$

Then the u_m, as functions in $C_c^\infty(\mathbb{R}^n)$, are a sequence of *exponentially good quasimodes* for H:

$$\|u_m\|_{L^2(\mathbb{R}^n)} \ge 1 - e^{-\lambda_m/C}, \quad \|(H - \lambda_m^2)u_m\|_{L^2(\mathbb{R}^n)} \le e^{-\lambda_m/C}, \quad \text{thus } \|\chi(H - \lambda_m^2)^{-1}\chi\|_{L^2\to L^2} \ge e^{\lambda_m/C}.$$

Polynomial resolvent estimates

Our goal in the remainder of the course is to prove that $\chi(H-\lambda^2)^{-1}\chi$ extends meromorphically to where $\chi(-\Delta-\lambda^2)^{-1}\chi$ extends holomorphically, and

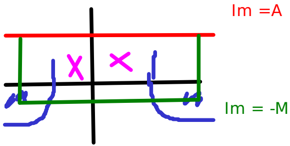
$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to L^2} \le |\lambda|^C,$$

when either $\operatorname{Im}\lambda \ge A$ or both $-\ln|\operatorname{Re}\lambda| \le C\operatorname{Im}\lambda$ and $|\operatorname{Re}\lambda|$ is large enough, for those H for which it holds. (We have already seen that $\|\chi(H-\lambda_m^2)^{-1}\chi\|_{L^2\to L^2} \ge e^{\lambda_m/C}$ occurs for certain H.) This is enough to show that, for n odd, the forward solution w to $(\partial_t^2+H)w = f \in C_c^\infty([T_0,T_1]\times\mathbb{R}^n)$ obeys

$$\chi w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}\chi(-\Delta+V-\lambda^2)^{-1}\chi\tilde f(\lambda,x)\,d\lambda, \qquad \tilde f(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)\,ds,$$

and, for any M,

$$\chi w(t,x) = -\sum_{\{\lambda_j:\ \operatorname{Im}\lambda_j > -M\}} iR_{\lambda_j}(t,x) + E(t,x),$$



with

$$|\chi E(t,x)| \le Ce^{-Mt},$$

and similar bounds on all derivatives of E. The proof [2] is just as in Part II, but taking advantage of the arbitrariness of N in the Paley–Wiener estimate $|\partial^\alpha \tilde f(\lambda,x)| \le C_{\alpha,N}e^{-T_0\operatorname{Im}\lambda}|\lambda|^{-N}$.

Semiclassical inversion

Returning now to the general problem, let

$$P = h^2 H = \sum_{j,k=1}^{n} a_{jk} hD_j hD_k + h \sum_{\ell=1}^{n} b_\ell hD_\ell + h^2 V, \qquad p = \sum_{j,k=1}^{n} a_{jk}\xi_j \xi_k \ge |\xi|^2/C,$$

where $h > 0$ is a (small) semiclassical parameter so that p is the semiclassical principal symbol of P. Then $p + 1$ is an elliptic symbol:

$$(1 + |\xi|^2)/C \le p + 1 \le C(1 + |\xi|^2),$$

and consequently $P + I$ is invertible for h small enough as in Hezari's Lecture 5, Slide 5. Indeed, let

$$\tilde{P} = (P+I)(-h^2\Delta+I)^{-1}, \quad \tilde{p} = \frac{p+1}{|\xi|^2+1}, \quad \mathrm{Op}(\tilde{p}^{-1})u(x) = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\cdot\xi/h}\tilde{p}(x,\xi)^{-1}u(y)dy\,d\xi,$$

so that $\tilde{p}$ is the semiclassical principal symbol of $\tilde{P}$, and $1/C \le \tilde{p} \le C$. Then

$$\left.\begin{aligned} \mathrm{Op}(\tilde{p}^{-1})\tilde{P} &= \mathrm{Op}(1) + O_{L^2\to L^2}(h) = I + O_{L^2\to L^2}(h) \\ \tilde{P}\,\mathrm{Op}(\tilde{p}^{-1}) &= \mathrm{Op}(1) + O_{L^2\to L^2}(h) = I + O_{L^2\to L^2}(h) \end{aligned}\right\} \implies \tilde{P}^{-1} \text{ exists and } \|\tilde{P}^{-1}\|_{L^2\to L^2} \le C.$$

Resolvent mapping properties

Thus, for $h > 0$ small enough, with

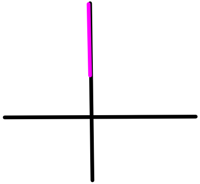
$$P = h^2 H = \sum_{j,k=1}^{n} a_{jk} hD_j hD_k + h \sum_{\ell=1}^{n} b_\ell hD_\ell + h^2 V, \qquad \tilde{P} = (P + I)(-h^2\Delta + I)^{-1},$$

we have established that $\tilde{P}$ is bijective $L^2 \to L^2$.

Combining this with the fact that $-h^2\Delta + I$ is bijective $H^2 \to L^2$, we conclude that $P + I$, and hence $H + h^{-2}$, are also bijective $H^2 \to L^2$. In other words the resolvent

$$R(\lambda) = (H - \lambda^2)^{-1},$$

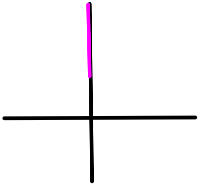
lambda = i/h



exists and is bijective $L^2 \to H^2$ when[3] $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \gg 0$.

(Hence, if H is symmetric on $H^2(\mathbb{R}^n)$, then it is self-adjoint and semi-bounded.)[4]

Openness of the resolvent set



Thus the resolvent

$$R(\lambda) = (H - \lambda^2)^{-1},$$

exists when $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \gg 0$.

Using the standard resolvent identity, as in Hassannezhad's Exercise 1.7,

$$R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2) R(\lambda) R(\lambda_0),$$

we can write

$$R(\lambda)(I + (\lambda_0^2 - \lambda^2) R(\lambda_0)) = R(\lambda_0),$$

and use a geometric series to show there is a neighborhood[5] of this ray where $R(\lambda)$ exists.

To go further, we use an elaboration of this resolvent identity due to Vodev. We will write

$$R(\lambda)(I + K(\lambda, \lambda_0)) = F(\lambda, \lambda_0),$$

where $\lambda \mapsto K(\lambda, \lambda_0)$ and $\lambda \mapsto F(\lambda, \lambda_0)$ are both holomorphic families of operators, with K being compact, so that the analytic Fredholm theorem is applicable.

Resolvent identity

Denote the resolvents

$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1}, \qquad R(\lambda) = (H - \lambda^2)^{-1},$$

and let $\lambda_0,\ \lambda$ be such that both resolvents exist at both points. Thus 0 subscripts denote objects we already understand.

Let $\chi_1 \in C_c^\infty(\mathbb{R}^n)$ be 1 near the set where $H \neq -\Delta$, and write

$$\begin{aligned} R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2) R(\lambda) R(\lambda_0) \\ &= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1 R(\lambda_0) + R(\lambda)(1-\chi_1)R(\lambda_0)\Big). \end{aligned} \tag{$*$}$$

The first term is already of the form we want to be able to write

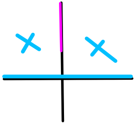
$$R(\lambda)(I + K(\lambda, \lambda_0)) = F(\lambda, \lambda_0),$$

with K compact, because $\chi_1 R(\lambda_0)$ is compact (recall that the resolvent maps $L^2 \to H^2$).

The second term needs some more work because $(1-\chi_1)R(\lambda_0)$ is not compact; we will use the fact that $(1-\chi_1)H = -(1-\chi_1)\Delta$ to write this term in terms of $R_0(\lambda)$.

To bring the second term of

$$\begin{aligned} R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2) R(\lambda) R(\lambda_0) \\ &= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1 R(\lambda_0) + R(\lambda)(1-\chi_1) R(\lambda_0)\Big). \end{aligned} \qquad (*)$$



to a suitable form, we write it in terms of $R_0(\lambda)$ using

$$\begin{aligned} R(\lambda)(1-\chi_1) &= R(\lambda)(1-\chi_1)(-\Delta - \lambda^2) R_0(\lambda) \\ &= R(\lambda)(-\Delta - \lambda^2)(1-\chi_1) R_0(\lambda) + R(\lambda)[-\Delta, \chi_1] R_0(\lambda) \qquad (**) \\ \text{Id} {=} &= \{(1-\chi_1) + R(\lambda)[-\Delta, \chi_1]\} R_0(\lambda). \end{aligned}$$

Inserting $(**)$ into $(*)$ gives

$$R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1 R(\lambda_0) + \{(1-\chi_1) + R(\lambda)[-\Delta, \chi_1]\} R_0(\lambda) R(\lambda_0)\Big).$$

Bringing the $R(\lambda)$ terms to the left, the other terms to the right, and factoring, gives

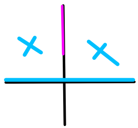
$$R(\lambda)(I + K(\lambda, \lambda_0)) = F(\lambda, \lambda_0), \qquad \text{where}$$

$$K(\lambda, \lambda_0) = (\lambda_0^2 - \lambda^2)\Big(\chi_1 R(\lambda_0) + [-\Delta, \chi_1] R_0(\lambda) R(\lambda_0)\Big), \; F(\lambda, \lambda_0) = R(\lambda_0) + (\lambda^2 - \lambda_0^2)(1-\chi_1) R_0(\lambda) R(\lambda_0).$$

Note that $\lambda \mapsto K(\lambda, \lambda_0)$ and $\lambda \mapsto F(\lambda, \lambda_0)$ are both holomorphic families of operators, with K being compact, so that the analytic Fredholm theorem is applicable.

Meromorphic continuation of the resolvent to the upper half plane

Thus, with



$$R_0(\lambda) = (-\Delta - \lambda^2)^{-1}, \qquad R(\lambda) = (H - \lambda^2)^{-1},$$

we have obtained

$$R(\lambda)(I + K(\lambda, \lambda_0)) = F(\lambda, \lambda_0),$$

where

$$K(\lambda, \lambda_0) = (\lambda_0^2 - \lambda^2)\Big(\chi_1 R(\lambda_0) + [-\Delta, \chi_1] R_0(\lambda) R(\lambda_0)\Big),$$
$$F(\lambda, \lambda_0) = R(\lambda_0) + (\lambda^2 - \lambda_0^2)(1 - \chi_1) R_0(\lambda) R(\lambda_0),$$

for λ and λ_0 both pure imaginary and sufficiently large (that is where we have defined the resolvent R). By the analytic Fredholm theorem, $\lambda \mapsto (I + K(\lambda, \lambda_0))^{-1}$ is meromorphic in the upper half plane, since K is compact and the inverse exists when $\lambda = \lambda_0$. Thus we have constructed the resolvent

$$R(\lambda) = F(\lambda, \lambda_0)(I + K(\lambda, \lambda_0))^{-1},$$

as a meromorphic family of operators in the upper half plane.

Next we will investigate pole-free regions and estimates.

Semiclassical estimates away from the real axis I

We consider $(H-\lambda^2)^{-1}$ when $\operatorname{Im}\lambda \gg 0$. We will show, for A large enough, the a priori estimate

$$\|u\|_{L^2} \leq \frac{C}{|\lambda|}\|(H-\lambda^2)u\|_{L^2}, \qquad \text{when} \qquad \operatorname{Im}\lambda \geq A. \tag{$*$}$$

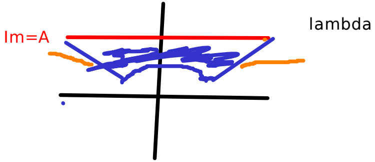
Since we already know that $(H-\lambda^2)^{-1}$ is meromorphic, that will imply it has no poles when $\operatorname{Im}\lambda \geq A$ and

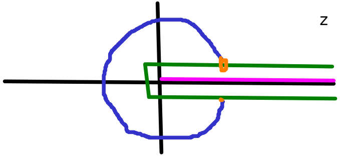
$$\|(H-\lambda^2)^{-1}\|_{L^2\to L^2} \leq \frac{C}{|\lambda|}, \qquad \text{when} \qquad \operatorname{Im}\lambda \geq A.$$

We will again use semiclassical estimates and prove $(*)$ by proving there is a constant C_0 such that

$$|z|=1 \quad \text{and} \quad \operatorname{dist}(z,[0,\infty)) \geq C_0 h \quad \Longrightarrow \quad \|u\|_{L^2} \leq \frac{C}{h}\|(P-z)u\|_{L^2}, \quad \text{where} \quad P=h^2H, \quad (**)$$

and apply this with $h=|\lambda|^{-1}$, $z=h^2\lambda^2$; if $\pm\operatorname{Re}\lambda \gg 0$, then $\operatorname{Im} z = 2h^2\operatorname{Re}\lambda\operatorname{Im}\lambda \approx \pm 2h\operatorname{Im}\lambda$. approx C_0 h





Semiclassical estimates away from the real axis II

We will prove that there is a constant C_0 such that

$$|z| = 1 \quad \text{and} \quad \operatorname{dist}(z, [0,\infty)) \geq C_0 h \quad \Longrightarrow \quad \|u\|_{L^2} \leq \frac{C}{h}\|(P-z)u\|_{L^2}, \qquad (**)$$

using the fact that we 'almost' have $P \geq 0$. We will consider real and imaginary parts separately.

Recall that we defined $\tilde{P} = (P+I)(-h^2\Delta + I)^{-1}$ and found that, uniformly for $h > 0$ small enough,

$$\|\tilde{P}\|_{L^2 \to L^2} \leq C, \qquad \text{and} \qquad \|\tilde{P}^{-1}\|_{L^2 \to L^2} \leq C.$$

Hence we use a semiclassical Sobolev norm and write $\|u\|_{H_h^2} := \|(-h^2\Delta + I)u\|_{L^2}$, so that

$$\|P + I\|_{H_h^2 \to L^2} \leq C, \qquad \text{and} \qquad \|(P+I)^{-1}\|_{L^2 \to H_h^2} \leq C.$$

Next, since $P = \sum_{j,k=1}^n a_{jk} hD_j hD_k + h\sum_{\ell=1}^n b_\ell hD_\ell + h^2 V$, and $\sum_{j,k=1}^n a_{jk}\xi_j\xi_k \geq |\xi|^2/C$, we get

$$\left\langle \sum_{j,k=1}^n a_{jk} hD_j hD_k u, u \right\rangle_{L^2} = \sum_{j,k=1}^n \langle a_{jk} hD_j u, hD_k u\rangle_{L^2} - h \sum_{j,k=1}^n (D_k a_{jk})\langle hD_j u, u\rangle_{L^2},$$

and consequently, using also $|\langle hD_j u, u\rangle_{L^2}| \leq \|hD_j u\|_{L^2}^2 + \|u\|_{L^2}^2$, we get

$$|\operatorname{Im}\langle Pu, u\rangle_{L^2}| \leq Ch(\|h\nabla u\|_{L^2}^2 + \|u\|_{L^2}^2) \leq C_1 h \|u\|_{H_h^2}^2.$$

Semiclassical estimates away from the real axis III

We are proving that there is a constant C_0 such that

$$|z| = 1 \quad \text{and} \quad \operatorname{dist}(z, [0, \infty)) \geq C_0 h \qquad \Longrightarrow \qquad \|u\|_{L^2} \leq \frac{C}{h} \|(P - z)u\|_{L^2}. \qquad (**)$$

We have shown that, with $\|u\|_{H^2_h} := \|(-h^2\Delta + I)u\|_{L^2}$, we have

$$\|P + I\|_{H^2_h \to L^2} \leq C, \qquad \text{and} \qquad \|(P + I)^{-1}\|_{L^2 \to H^2_h} \leq C, \qquad \text{and} \qquad |\operatorname{Im}\langle Pu, u\rangle_{L^2}| \leq C_1 h \|u\|_{H^2_h}.$$

The last implies

$$\operatorname{Im}\langle (P - z)u, u\rangle_{L^2} \leq C_1 h \|u\|^2_{H^2_h} - \operatorname{Im} z \|u\|^2_{L^2},$$

which we rewrite as

ab <= (1/2e)a^2 + (e/2)b^2

$$\operatorname{Im} z \|u\|^2_{L^2} \leq C_1 h \|u\|^2_{H^2_h} + \|(P - z)u\|_{L^2} \|u\|_{L^2}$$

Next use

(P+I)^{-1}(P+I)u

$$\|u\|_{H^2_h} \leq \|(P + I)^{-1}(P - z)u\|_{H^2_h} + \|(P + I)^{-1}(z + 1)u\|_{H^2_h} \leq C\|(P - z)u\|_{L^2} + C_3\|u\|_{L^2}.$$

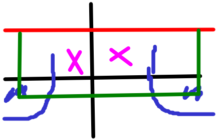
C_3

That leads to

$$(\operatorname{Im} z - C_2 h)\|u\|^2_{L^2} \leq \frac{C}{h} \|(P - z)u\|^2_{L^2},$$

which gives $(**)$ when $\operatorname{Im} z \geq C_0 h$. Arguing similarly with $-\operatorname{Im} z$ and $-\operatorname{Re} z$ gives the rest of $(**)$.[6]

Solution to the wave equation



Thus we have shown, with $R(\lambda) = (H - \lambda^2)^{-1}$, that

$$\|R(\lambda)\| \leq \frac{C}{|\lambda|}, \qquad \text{when} \qquad \operatorname{Im}\lambda \geq A.$$

This shows that

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda} R(\lambda)\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int e^{is\lambda} f(s,x)ds,$$

solves $(\partial_t^2 + H)w(t,x) = f(t,x)$, with $w(t,x) = 0$ when $t \ll 0$.

To study asymptotics as $t \to \infty$, we will deform the contour into the lower half plane. We first establish meromorphic continuation of the resolvent, using an elaboration of the above resolvent identity due to Vodev.

Vodev's identity I

Our previous resolvent identity

$$R(\lambda)(I + K(\lambda,\lambda_0)) = F(\lambda,\lambda_0), \qquad \text{where} \qquad R_0(\lambda) = (-\Delta - \lambda^2)^{-1}, \qquad R(\lambda) = (H - \lambda^2)^{-1},$$

$$K(\lambda,\lambda_0) = (\lambda_0^2 - \lambda^2)\Big(\chi_1 R(\lambda_0) + [-\Delta,\chi_1]R_0(\lambda)R(\lambda_0)\Big), \; F(\lambda,\lambda_0) = R(\lambda_0) + (\lambda^2 - \lambda_0^2)(1-\chi_1)R_0(\lambda)R(\lambda_0),$$

is not suited to continuation of the cutoff resolvent because multiplying K and F on the left and right by $\chi \in C_c^\infty(\mathbb{R}^n)$ leads to factors of $\chi R_0(\lambda)R(\lambda_0)\chi$. To continue across the real axis we need $R_0(\lambda)$ to be multiplied by χ on both sides.
Accordingly, we go back to

$$\begin{aligned} R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2)R(\lambda)R(\lambda_0) \\ &= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1 R(\lambda_0) + R(\lambda)(1-\chi_1)R(\lambda_0)\Big), \end{aligned}$$

and instead write

(1-x)^2 = 1 - 2x + x^2

1 = x(2-x) +(1-x)(1-x)

$$\begin{aligned} R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2)R(\lambda)R(\lambda_0) \\ &= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + R(\lambda)(1-\chi_1)(1-\chi_1)R(\lambda_0)\Big). \end{aligned}$$

The first term works just as well as before, and now the second term has an extra factor of $(1-\chi_1)$ which will be useful.

Vodev's identity II

Now proceed from

$$R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2)R(\lambda)R(\lambda_0)$$
$$= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + R(\lambda)(1-\chi_1)(1-\chi_1)R(\lambda_0)\Big).$$

and plug in the same identity from before

$$R(\lambda)(1-\chi_1) = \{(1-\chi_1) + R(\lambda)[-\Delta, \chi_1]\}R_0(\lambda)$$

but this time also use the analogous identity[7]

$$(1-\chi_1)R(\lambda_0) = R_0(\lambda_0)\{(1-\chi_1) + [-\Delta, \chi_1]R(\lambda_0)\},$$

to obtain

$$R(\lambda) - R(\lambda_0) = (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) +$$
$$\{(1-\chi_1) + R(\lambda)[-\Delta, \chi_1]\}R_0(\lambda)R_0(\lambda_0)\{(1-\chi_1) + [-\Delta, \chi_1]R(\lambda_0)\}\Big).$$

Now the only instance of two adjacent resolvents is $R_0(\lambda)R_0(\lambda_0)$, and there we use our original resolvent identity

$$(\lambda^2 - \lambda_0^2)R_0(\lambda)R_0(\lambda_0) = R_0(\lambda) - R_0(\lambda_0).$$

Vodev's identity III

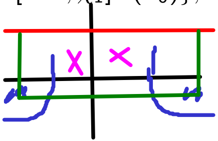
$$
\begin{aligned}
R(\lambda) - R(\lambda_0) &= (\lambda^2 - \lambda_0^2) R(\lambda) R(\lambda_0) \\
&= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + R(\lambda)(1-\chi_1)(1-\chi_1)R(\lambda_0) \Big) \\
&= (\lambda^2 - \lambda_0^2)\Big(R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + \\
&\qquad \{(1-\chi_1) + R(\lambda)[-\Delta,\chi_1]\} R_0(\lambda) R_0(\lambda_0) \{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\} \Big) \\
&= (\lambda^2 - \lambda_0^2) R(\lambda)\chi_1(2-\chi_1)R(\lambda_0) + \\
&\qquad \{(1-\chi_1) + R(\lambda)[-\Delta,\chi_1]\} \Big(R_0(\lambda) - R_0(\lambda_0) \Big) \{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\}.
\end{aligned}
$$

Bringing the $R(\lambda)$ terms to the left, the other terms to the right, and factoring, gives

$$R(\lambda)(I + K(\lambda,\lambda_0)) = F(\lambda,\lambda_0), \qquad \text{where}$$

$$K(\lambda,\lambda_0) = (\lambda_0^2 - \lambda^2)\chi_1(2-\chi_1)R(\lambda_0) + [-\Delta,\chi_1]\Big(R_0(\lambda) - R_0(\lambda_0)\Big)\{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\},$$

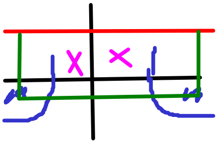
$$F(\lambda,\lambda_0) = R(\lambda_0) + (1-\chi_1)\Big(R_0(\lambda) - R_0(\lambda_0)\Big)\{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\}.$$



Applying the analytic Fredholm theorem again gives (for $\operatorname{Im}\lambda > 0$)

$$R(\lambda) = F(\lambda,\lambda_0)(I + K(\lambda,\lambda_0))^{-1}.$$

Meromorphic continuation of the resolvent to the lower half plane



Thus we have, for $\operatorname{Im}\lambda,\ \operatorname{Im}\lambda_0 > 0$, away from any poles,

$$R(\lambda) = F(\lambda,\lambda_0)(I + K(\lambda,\lambda_0))^{-1},$$

$$F(\lambda,\lambda_0) = R(\lambda_0) + (1-\chi_1)\Big(R_0(\lambda) - R_0(\lambda_0)\Big)\{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\},$$

$$K(\lambda,\lambda_0) = (\lambda_0^2 - \lambda^2)\chi_1(2-\chi_1)R(\lambda_0) + [-\Delta,\chi_1]\Big(R_0(\lambda) - R_0(\lambda_0)\Big)\{(1-\chi_1) + [-\Delta,\chi_1]R(\lambda_0)\},$$

take $\chi \in C_c^\infty(\mathbb{R}^n)$ such that $\chi\chi_1 = \chi_1$, and multiply on the left and right to obtain

$$\chi R\chi = \chi F(I+K)^{-1}\chi = \chi F\chi(I + K\chi)^{-1}, \qquad (*)$$

where we justify the second equality in two steps: 1) if λ_0 is fixed and λ is sufficiently close to λ_0, then $\|K(\lambda,\lambda_0)\|_{L^2\to L^2} < 1$ and we can use a convergent geometric series and $\chi K = K$ to write

$$(I+K)^{-1}\chi = (I - K + K^2 - \cdots)\chi = \chi(I - K\chi + K\chi K\chi - \cdots) = \chi(I + K\chi)^{-1},$$

and 2) use holomorphic continuation to extend to all $\operatorname{Im}\lambda,\ \operatorname{Im}\lambda_0 > 0$, away from any poles.

Finally the right side of $(*)$ continues meromorphically, to $\mathbb{C}$ if n is odd and to the logarithmic Riemann surface is n is even, so the left side does too.

The wave equation and resolvent for stronger perturbations

Let

$$H = \sum_{j,k=1}^{n} a_{jk} D_j D_k + \sum_{\ell=1}^{n} b_\ell D_\ell + V,$$

where all the b_ℓ, V are in $C_c^\infty(\mathbb{R}^n)$, and the a_{jk} are C^∞, real valued, and equal the identity matrix outside of a compact set, and $\sum_{j,k=1}^n a_{jk}\xi_j\xi_k \geq |\xi|^2/C$.

For $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$, we showed the forward solution of the wave equation

$$(\partial_t^2 + H)w(t,x) = f(t,x),$$

is given, for A large enough, by

$$w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds,$$

where the resolvent $(H-\lambda^2)^{-1}$ has the following properties:

- It is a meromorphic family of bounded operators[8] on L^2 for $\operatorname{Im}\lambda > 0$.
- It obeys $\|(H-\lambda^2)^{-1}\|_{L^2\to L^2} \leq C/|\lambda|$ when $\operatorname{Im}\lambda \geq A$.
- For any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(H-\lambda^2)^{-1}\chi$ continues meromorphically, to $\mathbb{C}$ if n is odd and to the logarithmic Riemann surface is n is even.

References

The meromorphic continuation of the resolvent is based on the black box method of Sjöstrand and Zworski: see Section 4.2 of Dyatlov and Zworski's book. Vodev's identity comes from Section 5 of his 2014 Math. Nachr. paper "Semi-classical resolvent estimates and regions free of resonances". The presentation here is based on Lemmas 2.1 and 2.2 of "Wave asymptotics for waveguides and manifolds with infinite cylindrical ends" by Christiansen and Datchev.

Exercises.

The exercises marked with a * are more central to the course. (They are not the more difficult ones.)

1. Construct a sequence of eigenfunctions of $\mathbb{S}^2$,

$$-\Delta_{\mathbb{S}^2}\varphi_m = \lambda_m^2\varphi_m, \qquad \text{with} \qquad \lambda_m \to \infty, \qquad \|\varphi_m\|_{L^2(\mathbb{S}^2)} = 1,$$

and such that there is a nonempty open $U \subset \mathbb{S}^2$ and a constant C such that

$$\|\varphi_m\|_{L^2(U)} \le e^{-\lambda_m/C}, \qquad \text{for every } m.$$

Hint: Let $\psi_m(x,y,z) = (x+iy)^m$, and use spherical coordinates to show that each ψ_m, restricted to the unit sphere $x^2+y^2+z^2=1$, is an eigenfunction.

2. * Complete the proof of wave decay outlined in the slide titled ‘Polynomial resolvent estimates’.

3. * Use semiclassical inversion to show that for any $\theta \in (0,\pi)$, $R(\lambda)$ exists and maps $L^2 \to H^2$, for $|\lambda|$ large enough when $\arg\lambda = \theta$.

4. Show that if H is symmetric on $H^2(\mathbb{R}^n)$, then it is self-adjoint and semi-bounded.

Hint: Use the criterion for self-adjointness which says that if H is symmetric and $H - a$ is surjective for some real a, then H is self-adjoint. This is part of Theorem 3.29 of Borthwick's *Spectral Theory*, and is similar to Hassannezhad's Section 1.4. Semi-boundedness follows from the fact that $R(\lambda)$ exists when $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \gg 0$. You may also enjoy adapting the result of this exercise to the operator $H = -c(x)^2\Delta$ by using $L^2(\mathbb{R}^n)$ with respect to $c(x)^{-2}dx$ as your Hilbert space.

5.* Show that $\|R(\lambda)\|_{L^2\to L^2} \le C/|\lambda|^2$ when $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \gg 0$. Use this and the standard resolvent identity

$$R(\lambda)(I + (\lambda_0^2 - \lambda^2)R(\lambda_0)) = R(\lambda_0)$$

to find a specific open neighborhood of this ray, in terms of C, where $R(\lambda)$ exists.

6.* Prove the there is a constant C_0 such that

$$\operatorname{Im} z \le -C_0 h \qquad \Longrightarrow \qquad \|u\|_{L^2} \le \frac{C}{h}\|(P - z)u\|_{L^2}.$$

and

$$\operatorname{Re} z \le -C_0 h \qquad \Longrightarrow \qquad \|u\|_{L^2} \le \frac{C}{h}\|(P - z)u\|_{L^2}.$$