Complex Scaling and Resonance Free Regions

Scattering Theory IV

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The wave equation and resolvent for stronger perturbations

Let

$$H = \sum_{j,k=1}^{n} a_{jk} D_j D_k + \sum_{\ell=1}^{n} b_\ell D_\ell + V,$$

where all the b_ℓ, V are in $C_c^\infty(\mathbb{R}^n)$, and the a_{jk} are C^∞, real valued, and equal the identity matrix outside of a compact set, and $\sum_{j,k=1}^n a_{jk}\xi_j\xi_k \geq |\xi|^2/C$.

For $f \in C_c^\infty([T_0, T_1] \times \mathbb{R}^n)$, we showed that the forward solution of the wave equation

$$(\partial_t^2 + H)w(t,x) = f(t,x),$$

is given, for A large enough, by

$$w(t,x) = \frac{1}{2\pi} \int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}(H-\lambda^2)^{-1}\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda} f(s,x)ds,$$

where the resolvent $(H-\lambda^2)^{-1}$ has the following properties:

- It is a meromorphic family of mapping $H^r(\mathbb{R}^n) \to H^{r+2}(\mathbb{R}^n)$ for any real r and for $\operatorname{Im}\lambda > 0$.
- It obeys $\|(H-\lambda^2)^{-1}\|_{L^2\to L^2} \leq C/|\lambda|$ when $\operatorname{Im}\lambda \geq A$.
- For any $\chi \in C_c^\infty(\mathbb{R}^n)$, $\chi(H-\lambda^2)^{-1}\chi$ continues meromorphically, to $\mathbb{C}$ if n is odd and to the logarithmic Riemann surface is n is even.

Polynomial resolvent estimates

Our goal in the remainder of the course is to prove

$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to L^2} \le |\lambda|^C,$$

when $-\ln|\operatorname{Re}\lambda| \le C\operatorname{Im}\lambda$ and $|\operatorname{Re}\lambda|$ is large enough, for those H for which it holds. (We have already seen in Part III that $\|\chi(H-\lambda_m^2)^{-1}\chi\|_{L^2\to L^2} \ge e^{\lambda_m/C}$ is possible for certain H.) This is enough to prove that, if n is odd, the forward solution w to $(\partial_t^2+H)w = f \in C_c^\infty([T_0,T_1]\times\mathbb{R}^n)$ obeys

$$\chi w(t,x) = \frac{1}{2\pi}\int_{-\infty+iA}^{\infty+iA} e^{-it\lambda}\chi(-\Delta+V-\lambda^2)^{-1}\chi\tilde{f}(\lambda,x)d\lambda, \qquad \tilde{f}(\lambda,x) = \int_{T_0}^{T_1} e^{is\lambda}f(s,x)ds,$$

and, for any M,

$$\chi w(t,x) = -\sum_{\{\lambda_j:\ \operatorname{Im}\lambda_j>-M\}} iR_{\lambda_j}(t,x) + E(t,x),$$

with

$$|\chi E(t,x)| \le Ce^{-Mt},$$

and similar bounds on all derivatives of E. The proof [1] is just as in Part II, but taking advantage of the arbitrariness of N in the Paley–Wiener estimate $|\partial^\alpha\tilde{f}(\lambda,x)| \le C_{\alpha,N}e^{-T_0\operatorname{Im}\lambda}|\lambda|^{-N}$.

Complex scaling in one dimension

To understand the behavior of waves as $t \to \infty$, we will use a contour deformation into the lower half plane. To get good control of the resolvent along the way, we will use *complex scaling*, which we now introduce, beginning with the simpler case of dimension $n = 1$.

This technique consists of holomorphically extending the operator H to an operator H_Γ which has better ellipticity properties, where $\Gamma = \{x + i\gamma(x) \colon \mathbb{R}\} \subset \mathbb{C}$, with $\gamma \in C^\infty(\mathbb{R})$ vanishing near $[-R, R]$, with R chosen such that $H = -\frac{d^2}{dx^2}$ away from $[-R, R]$, and with $\operatorname{sgn}\gamma''(x)\operatorname{sgn}\gamma(x) \geq 0$.

We will show that if $\chi \in C_c^\infty([-R, R])$, then for any λ we have

$$\chi(H - \lambda^2)^{-1}\chi = \chi(H_\Gamma - \lambda^2)^{-1}\chi.$$

The complex scaled operator in one dimension

More specifically, to define H_Γ, start with H and then replace every instance of x with z, a complex variable ranging over $\mathbb{R} \cup \{z \in \mathbb{C} \colon |\operatorname{Re} z| > R\}$, where R is chosen such that $H = -\frac{d^2}{dx^2}$ away from $[-R, R]$. Then restrict the resulting differential operator to the curve $\Gamma = \{x + i\gamma(x) \colon \mathbb{R}\}$, with $\gamma \in C^\infty(\mathbb{R})$ a function which vanishes near $[-R, R]$ and obeys $\operatorname{sgn} \gamma''(x) \operatorname{sgn} \gamma(x) \geq 0$.

This means replacing every instance of $\frac{d}{dx}$ with $\frac{d}{dz}\big|_\Gamma = (1 + i\gamma'(x))^{-1} \frac{d}{dx}$, giving

$$H = -a(x)\frac{d^2}{dx^2} - ib(x)\frac{d}{dx} + V(x),$$
$$H_\Gamma = -\frac{a(x)}{1 + i\gamma'(x)}\frac{d}{dx}\frac{1}{1 + i\gamma'(x)}\frac{d}{dx} - \frac{ib(x)}{1 + i\gamma'(x)}\frac{d}{dx} + V(x).$$

One can check that the definition of $\frac{d}{dz}\big|_\Gamma$ is independent of choice of parametrization[2] but we will only need the fact that if u is complex differentiable at some $z = x + i\gamma(x) \in \Gamma$, then, by the chain rule,

$$\frac{d}{dz}\Big|_\Gamma u(z) = \frac{1}{1 + i\gamma'(x)}\frac{d}{dx}u(x + i\gamma(x)) = \frac{d}{dz}u(z).$$

Resolvent of the complex scaled operator in one dimension: semiclassical inversion

We have thus defined

$$H_\Gamma = -\frac{a(x)}{1+i\gamma'(x)}\frac{d}{dx}\frac{1}{1+i\gamma'(x)}\frac{d}{dx} - \frac{ib(x)}{1+i\gamma'(x)}\frac{d}{dx} + V(x).$$

Suppose now that $\gamma'(x) = \tan\theta$ when $|x| \geq R'$ for some $R' > R$ and $\theta \in (-\pi/2, \pi/2)$. Then, when $|x| \geq R'$, H_Γ equals

$$H_\theta := -(1 + i\tan\theta)^{-2}\frac{d^2}{dx^2},$$

which is a scalar multiple of the free Laplacian. We can now study the resolvent as in Part III. Let

$$P_\Gamma = h^2 H_\Gamma, \qquad p_\Gamma(x,\xi) = (1 + i\gamma'(x))^{-2}a(x)|\xi|^2,$$

so that

$$|\arg p_\Gamma| = |\arg(1 + i\gamma')^{-2}| \leq 2|\theta| < \pi,$$

and consequently $|p_\Gamma + 1| \geq (1 + |\xi|^2)/C$. By semiclassical elliptic inversion,

$$(P_\Gamma + I)^{-1} = h^{-2}(H_\Gamma + h^{-2})^{-1}$$

is bounded $L^2 \to H_h^2$ and we can define the resolvent[3]

$$R_\Gamma(\lambda) = (H_\Gamma - \lambda^2)^{-1} \colon L^2 \to H^2,$$

when $\operatorname{Re}\lambda = 0$ and $\operatorname{Im}\lambda \gg 0$.

Resolvent of the complex scaled operator in one dimension: meromorphic continuation I

We have thus defined

$$H_\Gamma = -\frac{a(x)}{1+i\gamma'(x)}\frac{d}{dx}\frac{1}{1+i\gamma'(x)}\frac{d}{dx} - \frac{ib(x)}{1+i\gamma'(x)}\frac{d}{dx} + V(x), \qquad H_\theta = -(1+i\tan\theta)^{-2}\frac{d^2}{dx^2}.$$

The corresponding resolvents

$$R_\Gamma(\lambda) = (H_\Gamma - \lambda^2)^{-1}, \qquad R_\theta(\lambda) = (H_\theta - \lambda^2)^{-1}, \qquad \text{mapping } L^2 \to H^2,$$

are defined when $\operatorname{Re}\lambda = 0$, $\operatorname{Im}\lambda \gg 0$, and

$$R_\theta(\lambda) = (1+i\tan\theta)^2(-\Delta - (1+i\tan\theta)^2\lambda^2)^{-1}$$

continues holomorphically to the set where $\operatorname{Im}(1+i\tan\theta)\lambda > 0$, i.e. $\arg\lambda \in (-\theta, \pi-\theta)$. We can now use a resolvent identity:

$$R_\Gamma(\lambda)(I + K(\lambda,\lambda_0)) = F(\lambda,\lambda_0),$$

where

$$K(\lambda,\lambda_0) = (\lambda_0^2 - \lambda^2)\Big(\chi_1 R_\Gamma(\lambda_0) + [-\Delta,\chi_1]R_\theta(\lambda)R_\Gamma(\lambda_0)\Big),$$

$$F(\lambda,\lambda_0) = R_\theta(\lambda_0) + (\lambda^2 - \lambda_0^2)(1-\chi_1)R_\theta(\lambda)R_\Gamma(\lambda_0),$$

as in Part III to show that $R_\Gamma(\lambda)$ continues meromorphically to the same set.[4]

Resolvent of the complex scaled operator in one dimension: meromorphic continuation II

Another way to prove meromorphic continuation of the resolvent of

$$H_\Gamma = -\frac{a(x)}{1+i\gamma'(x)}\frac{d}{dx}\frac{1}{1+i\gamma'(x)}\frac{d}{dx} - \frac{ib(x)}{1+i\gamma'(x)}\frac{d}{dx} + V(x).$$

is to use the ODE method of variation of parameters to construct its integral kernel. Let u_+, u_- solve

$$(H_\Gamma - \lambda^2)u_\pm = 0, \quad u_-(x) = e^{-(1+i\tan\theta)i\lambda x} \text{ for } x \le -R', \quad u_+(x) = e^{(1+i\tan\theta)i\lambda x} \text{ for } x \ge R'.$$

If $\arg\lambda \in (-\theta, \pi-\theta)$, then $u_- \in L^2(\mathbb{R}_-)$ and $u_+ \in L^2(\mathbb{R}_+)$, and the solution $u \in L^2(\mathbb{R})$ to

$$(H_\Gamma - \lambda^2)u = f, \qquad \text{where} \qquad f \in L^2(\mathbb{R}),$$

is given by

$$u(x) = u_-(x)\int_x^\infty I_+ + u_+(x)\int_{-\infty}^x I_-, \qquad \text{where} \qquad I_\pm = \frac{u_\pm(y)(1+i\gamma'(y))^2 f(y)dy}{a(y)W(y)},$$

where $W = u_-u_+' - u_+u_-'$, unless this Wronskian is zero; indeed first check this when f is compactly supported and $\operatorname{Re}\lambda$, $\operatorname{Im}\lambda \gg 0$, and extend to all $f \in L^2(\mathbb{R})$ and λ with $\arg\lambda \in (-\theta, \pi-\theta)$.

The resolvent exists everywhere except where $W = 0$, and since W is holomorphic in λ its zeros are discrete and all have finite order.

Equivalence of the complex scaled operator in one dimension

Having shown that $(H_\Gamma - \lambda^2)^{-1}$ is meromorphic in $\arg\lambda \in (-\theta, \pi - \theta)$, we now prove that

$$\chi(H - \lambda^2)^{-1}\chi = \chi(H_\Gamma - \lambda^2)^{-1}\chi,$$

for the same range of λ, provided $\chi \in C_c^\infty[-R, R]$. Fix $f \in L^2$, with $\operatorname{supp} f \subset [-R, R]$, and fix λ with $\operatorname{Re}\lambda = 0$, and $\operatorname{Im}\lambda \gg 0$. Suppose $u \in L^2$ solves

$$(H - \lambda^2)u = f.$$

When $\pm x > R$, the equation becomes $-u'' - \lambda^2 u = 0$, and since $u \in L^2$ that means

$$u(x) = C_{\pm,f,\lambda} e^{\pm i\lambda x}, \qquad \pm x \geq R.$$

We extend this to a function of a complex variable and restrict to Γ to obtain u_Γ given by

$$u_\Gamma(x) = u(x + i\gamma(x)), \quad \text{and solving} \quad (H_\Gamma - \lambda^2)u_\Gamma = f.$$

Since $\chi u_\Gamma = \chi u$, it is enough to check that $u_\Gamma \in L^2$. That follows from the fact that $e^{\pm i\lambda(x+i\gamma(x))}$ is a decaying exponential as $\pm x \to \infty$.

Semiclassical estimate for the complex scaled operator in one dimension I

Thus we have shown that $\chi(H-\lambda^2)^{-1}\chi = \chi(H_\Gamma - \lambda^2)^{-1}\chi$ for $\arg\lambda \in (-\theta, \pi-\theta)$, and also that $(H_\Gamma - \lambda^2)^{-1}$ is meromorphic there. We will prove that, if $\theta > 0$ is fixed small enough, then

$$\|\chi(H-\lambda^2)^{-1}\chi\|_{L^2\to L^2} \le |\lambda|^C,$$

when $-\ln|\operatorname{Re}\lambda| \le C \operatorname{Im}\lambda$ and $\operatorname{Re}\lambda$ is large enough, by proving the a priori estimate

$$\|u\|_{L^2} \le |\lambda|^C \|(H_\Gamma - \lambda^2)u\|_{L^2}.$$

We will deduce the latter from the semiclassical estimate

$$\|u\|_{L^2} \le h^{-C}\|(P_\Gamma - z)u\|_{L^2},$$

where $P_\Gamma = h^2 H_\Gamma$, for $|z-1| \le h\ln(1/h)/C$, applied with $h = |\operatorname{Re}\lambda|^{-1}$, $z = h^2\lambda^2$. Recall that if $\pm\operatorname{Re}\lambda \gg 0$, then $\operatorname{Im} z = 2h^2 \operatorname{Re}\lambda \operatorname{Im}\lambda \approx \pm 2h\operatorname{Im}\lambda$.

Semiclassical estimate for the complex scaled operator in one dimension II

To prove

$$\|u\|_{L^2} \leq h^{-C}\|(P_\Gamma - z)u\|_{L^2}, \qquad |z-1| \leq h\ln(1/h)/C, \qquad (*)$$

we will prove

$$\|u\|_{L^2} \leq C\varepsilon^{-1}\|e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h}u\|_{L^2}, \qquad (**)$$

where $\varepsilon = h\ln(1/h)$ and $Q = \mathrm{Op}_h(q)$, with $q \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ to be determined. (This is an instance of the PDE principle of carefully choosing the norms/spaces for our estimates.) To prove that $(**)$ implies $(*)$, we write

$$\|e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h}u\|_{L^2} \leq \|e^{\varepsilon Q/h}(P_\Gamma - z)e^{-\varepsilon Q/h}u\|_{L^2} + \|(z-1)u\|_{L^2},$$

which shows that $(**)$ implies

$$\|u\|_{L^2} \leq C\varepsilon^{-1}\|e^{\varepsilon Q/h}(P_\Gamma - z)e^{-\varepsilon Q/h}u\|_{L^2}, \qquad |z-1| \leq Mh\ln(1/h). \qquad (***)$$

Next $(***)$ implies $(*)$ because

$$\|e^{\varepsilon Q/h}\| = \Big\|\sum_{k=0}^\infty \frac{\varepsilon^k Q^k}{k!h^k}\Big\| \leq \sum_{k=0}^\infty \frac{\varepsilon^k\|Q\|^k}{k!h^k} = e^{\varepsilon\|Q\|/h} = e^{\|Q\|\ln(1/h)} = h^{-\|Q\|},$$

where all norms are $L^2 \to L^2$.

Semiclassical estimate for the complex scaled operator in one dimension III

It thus remains to prove

$$\|u\|_{L^2} \le C\varepsilon^{-1}\|e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h}u\|_{L^2}, \qquad (**)$$

where $\varepsilon = h\ln(1/h)$ and $Q = \mathrm{Op}_h(q)$. For this we will use the Taylor expansion

$$e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h} = (P_\Gamma - 1) + \varepsilon[Q/h, P_\Gamma] + O_{L^2\to L^2}(\varepsilon^2);$$

we will choose $q \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ so that $\varepsilon[Q/h, P_\Gamma]$ is semiclassically elliptic wherever $P_\Gamma - 1$ isn't, but let us first explain the $O_{L^2\to L^2}(\varepsilon^2)$ remainder. To do so we write[5]

$$\mathrm{ad}_B\, A = [B, A], \qquad e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h} = e^{\varepsilon\, \mathrm{ad}_{Q/h}}(P_\Gamma - 1),$$

and use the Taylor expansion with integral remainder:

$$\begin{aligned} e^{\varepsilon\, \mathrm{ad}_{Q/h}}(P_\Gamma - 1) =&(P_\Gamma - z) + \varepsilon(\mathrm{ad}_{Q/h})P_\Gamma \\ &+ \frac{1}{2}\varepsilon^2(\mathrm{ad}_{Q/h})^2 P_\Gamma + \cdots + \frac{1}{K!}\varepsilon^K(\mathrm{ad}_{Q/h})^K P_\Gamma \\ &+ \frac{1}{K!}\varepsilon^{K+1}\int_0^1 (1-t)^K e^{\varepsilon t\, \mathrm{ad}_Q/h}(\mathrm{ad}_{Q/h})^{K+1}P_\Gamma\, dt \end{aligned}$$

For any K, all terms in the second line are $O_{L^2\to L^2}(\varepsilon^2)$. If K is big enough, the integral remainder term is $O(\varepsilon^2)$ because $\|e^{\varepsilon Q/h}\|_{L^2\to L^2} \le h^{-\|Q\|_{L^2\to L^2}}$.

Semiclassical estimate for the complex scaled operator in one dimension IV

Thus we have

$$e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h} = (P_\Gamma - 1) + \varepsilon[Q/h, P_\Gamma] + O_{L^2 \to L^2}(\varepsilon^2),$$

and so to prove

$$\|u\|_{L^2} \le C\varepsilon^{-1}\|e^{\varepsilon Q/h}(P_\Gamma - 1)e^{-\varepsilon Q/h}u\|_{L^2}, \qquad (**)$$

it is enough to show

$$\|u\|_{L^2} \le C\varepsilon^{-1}\|Su\|_{L^2}, \qquad S := (P_\Gamma - 1) + \varepsilon[Q/h, P_\Gamma],$$

where $\varepsilon = h\ln(1/h)$, $Q = \mathrm{Op}(q)$, $q \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ to be determined. The principal symbol of S is

$$s = p_\Gamma - 1 + i\varepsilon\{p_\Gamma, q\} = \frac{a(x)\xi^2}{(1+i\gamma'(x))^2} - 1 + i\varepsilon\{p_\Gamma, q\} = a(x)\xi^2(1+O(\theta)) - 1 + O(\varepsilon).$$

Hence for $\theta > 0$ small enough and $h > 0$ small enough, we get

$$|s| \ge (1+\xi^2)/C, \qquad \text{when} \qquad \xi^2 \le \min a/2 \text{ or } \xi^2 \ge 2\max a.$$

Fix $\varphi \in C_c^\infty(0,\infty)$ which is 1 near $[\min a/2, 2\max a]$. Then, by the composition formula and L^2 boundedness (see Hezari Lecture 5),

$$\|\mathrm{Op}(1-\varphi(\xi^2))u\|_{L^2} \le C\|\mathrm{Op}((1-\varphi(\xi^2))s^{-1})Su\|_{L^2} + Ch\|u\|_{L^2} \le C\|Su\|_{L^2} + Ch\|u\|_{L^2}.$$

Semiclassical estimate for the complex scaled operator in one dimension V

Our goal now is to prove, with $\varepsilon = h\ln(1/h)$, $Q = \operatorname{Op}(q)$, $q \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ to be determined, that

$$\|u\|_{L^2} \le C\varepsilon^{-1}\|Su\|_{L^2}, \quad S = (P_\Gamma - 1) + \varepsilon[Q/h, P_\Gamma], \quad s = p_\Gamma - 1 + i\varepsilon\{p_\Gamma, q\}, \quad p_\Gamma = \frac{a(x)\xi^2}{(1+i\gamma'(x))^2},$$

and thus far we have shown, for $\varphi \in C_c^\infty(0,\infty)$ which is 1 near $[\min a/2, 2\max a]$,

$$\|\operatorname{Op}(1-\varphi(\xi^2))u\|_{L^2} \le C\|Su\|_{L^2} + Ch\|u\|_{L^2}. \qquad (\square)$$

We will choose $q \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ so as to get

$$\operatorname{Im} s \le -\varepsilon, \qquad \text{when} \qquad \xi^2 \in \operatorname{supp}\varphi. \qquad (\square\square)$$

That will imply, by the sharp Gårding inequality (see Hezari's Lecture 5)

$$\operatorname{Im}\langle (S+i\varepsilon)\operatorname{Op}(\varphi(\xi^2))u, \operatorname{Op}(\varphi(\xi^2))u\rangle_{L^2} \le Ch\|u\|_{L^2}^2,$$

and hence, since $\|[S, \operatorname{Op}(\varphi(\xi^2))]\|_{L^2\to L^2} \le Ch$,

$$\varepsilon\|\operatorname{Op}(\varphi(\xi^2))u\|_{L^2}^2 \le \|S\operatorname{Op}(\varphi(\xi^2))u\|_{L^2}\|\operatorname{Op}(\varphi(\xi^2))u\|_{L^2} + Ch\|u\|_{L^2}^2 \le \|Su\|_{L^2}\|\operatorname{Op}(\varphi(\xi^2))u\|_{L^2} + Ch\|u\|_{L^2}^2,$$

which implies

$$\varepsilon\|\operatorname{Op}(\varphi(\xi^2))u\|_{L^2}^2 \le C\varepsilon^{-1}\|Su\|_{L^2}^2 + Ch\|u\|_{L^2}^2.$$

Combining with $(\square)$ gives the result so it is enough to construct q so that $(\square\square)$ holds.

Escape function construction

Given $\varphi \in C_c^\infty(0,\infty)$, we now wish to construct a q such that

$$\operatorname{Im} s \leq -\varepsilon \quad \text{when} \quad \xi^2 \in \operatorname{supp}\varphi, \qquad \text{where} \quad s = p_\Gamma - 1 + i\varepsilon\{p_\Gamma, q\}, \quad p_\Gamma = \frac{a(x)\xi^2}{(1+i\gamma'(x))^2}. \quad (\square\square)$$

We have $\operatorname{Im} s = \operatorname{Im} p_\Gamma + \varepsilon\{\operatorname{Re} p_\Gamma, q\}$. When $|x| \geq R$, we have $\gamma'(x) = \tan\theta$ and hence

$$\operatorname{Im} s = \operatorname{Im} p_\Gamma + O(\varepsilon) = -2a(x)\xi^2 \tan\theta(1+\tan\theta^2)^{-2} + O(\varepsilon) \leq -1/C.$$

Since $\operatorname{Im} p_\Gamma \leq 0$ always, it is enough to construct q so that

$$-\{\operatorname{Re} p_\Gamma, q\} = \{q, \operatorname{Re} p_\Gamma\} \geq 1/C, \qquad \text{when } |x| \leq R \text{ and } \xi^2 \in \operatorname{supp}\varphi.$$

For that we take $q(x,\xi) = q_1(x)q_2(\xi)$ such that $q_2(\xi) = \xi$ near $\operatorname{supp}\varphi$ and write

$$\{q, \operatorname{Re} p_\Gamma\} = q_1(x)\partial_x \operatorname{Re} p_\Gamma - q_1'(x)\xi\partial_\xi \operatorname{Re} p_\Gamma = q_1(x)a'(x)\xi^2 + O(\theta) - 2q_1'(x)\operatorname{Re} p_\Gamma,$$

so it is enough to take q_1'/q_1 a large constant near $[-R,R]$. Such a q is called an *escape function* or *Lyapounov function*. Its key property is being monotonic (in this case increasing) along the flow of p.

Higher dimensions I

Much of the above discussion carries with minor changes to higher dimensions. To define the complex scaled operator we use polar coordinates, and replace ∂_r with $\partial_r|_\Gamma = (1 + i\gamma'(r))^{-1}\partial_r$.
Constructing the resolvent $(H_\Gamma - \lambda^2)^{-1}$ by semiclassical inversion, and using resolvent identities to meromorphically continue, works just as before. The same approch as before, but with separation of variables, shows

$$\chi(H_\Gamma - \lambda^2)^{-1}\chi = \chi(H - \lambda^2)^{-1}\chi.$$

As before, fix $f \in L^2$, $\operatorname{supp} f \subset \operatorname{supp}\chi$, and fix λ with $\operatorname{Re}\lambda = 0$, and $\operatorname{Im}\lambda \gg 0$. Suppose $u \in L^2$ solves

$$(H - \lambda^2)u = f.$$

When $|x|$ is large, the equation becomes $-\Delta u - \lambda^2 u = 0$, and writing $u(r,\theta) = \sum_{j=0}^\infty u_j(r)Y_j(\theta)$, where Y_j are spherical harmonics, $-\Delta_{\mathbb{S}^{n-1}}Y_j = \sigma_j^2 Y_j$, we get

$$-u_j'' - (n-1)r^{-1}u_j' + r^{-2}\sigma_j^2 u_j - \lambda^2 u_j = 0.$$

Since $u \in L^2$, by the WKB approximation $u_j(r) \sim C_j e^{i\lambda r} r^{(1-n)/2}$ (in fact, u_j is given by a Hankel function). Extend this to a function of a complex variable and restrict to Γ to obtain $u_{j,\Gamma}$ given by

$$u_{j,\Gamma}(r) = u(r + i\gamma(r)), \quad \text{and solving} \quad (H_\Gamma - \lambda^2)u_\Gamma = f.$$

Since $\chi u_\Gamma = \chi u$, it is enough to check that $u_\Gamma \in L^2$. That follows from the fact that $e^{i\lambda(r+i\gamma(r))}$ is a decaying exponential as $r \to \infty$.

Higher dimensions II

After that, the analysis is again the same up until we get to the point of wanting to construct $q \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$\{q, \operatorname{Re} p_\Gamma\} \geq 1/C, \qquad \text{when } (x,\xi) \in K, \qquad \text{where } p_\Gamma = \sum_{j,k=1}^{n} \frac{a_{j,k}(x)|\xi|^2}{(1 + i\gamma(|x|))^2}$$

where K is a given compact subset of $\{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \text{ such that } \xi \neq 0\}$. The existence of such a q implies that p is *nontrapping*, i.e. that all *bicharacteristics* (aka integral curves) of the vector field H_p given by $H_p f = \{p, f\}$ escape to infinity. Conversely, one can show that if p is nontrapping, then there is such a q.

Nontrapping escape functions

Let W be a neighborhood of K. For each $\rho \in K$, let γ_ρ be the bicharacteristic through ρ. Fix $a_\rho < 0 < b_\rho$ such that a_ρ is the max of the negative times for which $\gamma_\rho \notin W$ and b_ρ is the min of the positive times for which $\gamma_\rho \notin W$. Let Σ_ρ be a hypersurface in $\mathbb{R}^n \times \mathbb{R}^n$ transversal to γ_ρ at ρ. Let $\varphi_\rho \in C_c^\infty(\Sigma_\rho)$ be nonnegative, 1 near ρ, and supported in a sufficiently small neighborhood U_ρ of ρ, small enough that the map

$$\Phi_\rho \colon (a_\rho, b_\rho) \times U_\rho \to T^*M, \quad \Phi_\rho(t, y) = \exp(tH_p)y,$$

is a diffeomorphism onto its image. Then Φ_ρ defines product coordinates (t, y) on its image, and it makes sense to define

$$q_\rho(t, y) = \chi_\rho(t)\varphi_\rho(y),$$

where $\chi_\rho \in C_c^\infty((a_\rho, b_\rho))$ has $\chi'_\rho < 0$ on a sufficiently large subset of (a_ρ, b_ρ) that

$$\{p, q_\rho\} = H_p q_\rho = \chi'_\rho \varphi_\rho$$

is strictly negative on $\gamma_\rho \cap K$. It is then also strictly negative on a neighborhood of $\gamma_\rho \cap K$. Using the fact that such neighborhoods cover K, choose finitely many ρ_j for $j = 1, \ldots, J$ and put

$$q = \sum_{j=1}^{J} q_{\rho_j}.$$

References

The discussion of complex scaling is a variant of the one in Sjöstrand and Zworski's 1991 JAMS paper "Complex Scaling and the Distribution of Scattering Poles", but making more use of ODE methods. See sections 2.7 and 4.5 of Dyatlov and Zworski's book for more, and see also the note `http://www.math.purdue.edu/~kdatchev/res.ps` and pages 36 to 39 of Tang and Zworski's notes `https://math.berkeley.edu/~zworski/tz1.pdf` for additional short versions of the one-dimensional case. The semiclassical estimates follow Section 4 of Sjöstrand and Zworski's 2007 Duke paper "Fractal Upper Bounds on the Density of Semiclassical Resonances" and see also Section 6.4 of Dyatlov and Zworski; it is shown there that, keeping track of constants, one can get an optimal result from this proof.

Exercises.

The exercises marked with a * are more central to the course. (They are not the more difficult ones.)

1. * Complete the proof of wave decay outlined in the slide titled 'Polynomial resolvent estimates'.

2. Let $C \subset \mathbb{C}$ be a C^∞ curve. Let $c\colon \mathbb{R} \to \mathbb{C}$ be a parametrization of C. Prove that

$$\frac{d}{dz}\Big|_C f = (c'(t))^{-1} \frac{d}{dt} f(c(t))$$

is independent of the parametrization by showing that if $\tilde{c}$ is another parametrization, with $c(t_0) = \tilde{c}(s_0) = z_0$ for some $z_0 \in C$ and some real t_0 and s_0, then the quotient $\tilde{c}'(s_0)^{-1} c'(t_0)$ is real, so that

$$(c'(t_0))^{-1} \frac{d}{dt} f(c(t_0)) = (\tilde{c}'(s_0))^{-1} \frac{d}{ds} f(\tilde{c}(s_0)).$$

3. Take $\gamma \in C^\infty(\mathbb{R})$ vanishing near $[-R, R]$ and obeying $\operatorname{sgn} \gamma''(x) = \operatorname{sgn} \gamma(x)$. Let $\theta_1 = \arctan \inf \gamma'$ and $\theta_1 = \arctan \sup \gamma'$. For what range of values of φ can you use semiclassical elliptic inversion to construct

$$R_\Gamma(\lambda) = (H_\Gamma - \lambda^2)^{-1}\colon L^2 \to H^2,$$

when $\arg \lambda = \varphi$ and $|\lambda|$ is large enough? (More general contours are used in the numerical technique *Perfectly Matched Layers*: see for example `http://math.mit.edu/~stevenj/18.369/pml.pdf`)

4. * Use the techniques of Part III to prove that $(H_\Gamma - \lambda^2)^{-1}$ is meromorphic when $\arg \lambda \in (-\theta, \pi - \theta)$.

Hint: Mimic the proof that $R(\lambda)$ is meromorphic in the upper half plane, but with H_Γ in place of H (the operator we wish to understand) and with H_θ in place of $-\Delta$ (the operator we already understand).

5. Verify that if $Q = \mathrm{Op}(q)$, $q \in C_c^\infty$, and A is a differential operator with smooth coefficients, then

$$e^{\varepsilon Q} A e^{-\varepsilon Q} = e^{\varepsilon \, \mathrm{ad}_Q} A,$$

by checking that both sides solve the operator equation

$$\partial_\varepsilon B = [Q, B] = \mathrm{ad}_Q B,$$

with the same initial condition at $\varepsilon = 0$.