How much detail should go into a proof?

We first examine what proofs are for, and then the role of detail. The best teacher on this is Gottlob Frege, and we get the answer from the opening sentences of his 1879 booklet *Concept Writing*:

> In discovering a scientific truth we pass, as a rule, through several degrees of certainty. Perhaps first conjectured on the basis of an insufficient number of particular cases, a general proposition comes to be more and more securely established by being connected with other truths through chains of inferences, whether consequences are derived from it that are confirmed in some other way or whether, conversely, it is seen to be a consequence of propositions already established.

Thus, the purpose of a proof is to establish chains of inferences which deduce propositions from one other. The purpose of adding detail is to make the gaps in those chains smaller, and ultimately to close the gaps completely. Frege's 'concept writing' is a logical notation which makes this possible. He writes:

> I had to bend every effort to keep the chain of inferences free of gaps. In attempting to comply with this requirement in the strictest possible way I found the inadequacy of language to be an obstacle. No matter how unwieldy the expressions I was ready to accept, I was less and less able, as the relations became more and more complex, to attain the precision that my purpose required. This deficiency led me to the idea of the present concept writing. Its first purpose, therefore, is to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated.

Frege's concept writing is the predecessor of the modern logical notation in which we write the definition of continuity, for instance, as follows:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall y \quad (|x - y| < \delta \rightarrow |f(x) - f(y)| < \varepsilon).$$

(Frege's notation has advantages over ours, including a beautiful layout – see below!) As he explains, other such notations are used in other fields, including algebra (actually, the example above contains algebraic as well as logical notation) and chemistry. He compares his concept writing to a microscope:

> I believe that I can best make the relation of my concept writing to ordinary language clear if I compare it to the relation of the microscope to the eye. In its range of possible uses and in the versatility with which it can adapt to the most diverse circumstances, the eye has a great advantage over the microscope. Considered as an optical instrument, to be sure, it has many imperfections, which ordinarily remain unnoticed on account of its intimate connection with our mental life. But, as soon as scientific goals demand great sharpness of resolution, the eye proves to be inadequate. The microscope, on the other hand, is perfectly suited to precisely such goals, but that is just why it is unsuited for all others.

Thus, logical notation, such as Frege's or its modern successor, allows us to zoom into the smallest gaps in a chain of inferences. But precisely because of this high level of zoom, it prevents us from seeing the big picture. For that we must stay more zoomed out, and accordingly tolerate bigger gaps in the chain of inferences which are only to be closed as results arrive at their final form.

Now to relate this to the practical question of how much detail to include. Putting results into their final form is not part of most people's program, so we adapt the sizes of gaps we leave in our proofs to the level at which we are working. We bear in mind the appropriate balance between the big picture and the fine detail. We prioritize revealing the most important connections between propositions, with the more major and more minor components of the argument having proportionally more major and more minor roles.

For instance, consider the proposition that 2 has a positive real square root. This follows from the fact that 1 and 4 have positive square roots, together with continuity of the function $f(x) = x^2$ and completeness of the real numbers. On a big picture level, this finishes the proof: see Figure 1.

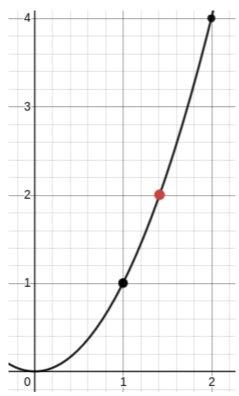


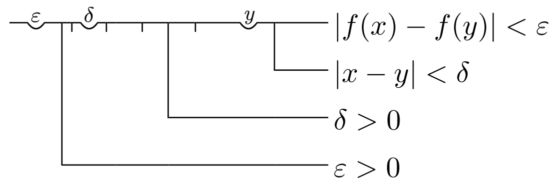
FIGURE 1. As we move along the graph of $y = x^2$ from $(1, 1)$ to $(2, 4)$, by continuity the y value must pass through 2. By completeness of the real numbers, there must be a corresponding x value. This x value is the square root of 2.

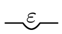
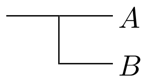
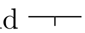
In more detail, we bring in the continuity and completeness using the intermediate value theorem, which says that if f is continuous, $a < b$, and $f(a) < m < f(b)$, then there is $c \in (a, b)$ such that $f(c) = m$. We apply the intermediate value theorem with $a = 1$, $b = 2$, and $m = 2$.

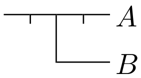
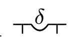
In yet more detail, $f(x) = x^2$ is continuous because $g(x) = x$ is continuous and products of continuous functions are continuous. The intermediate value theorem can be proved as follows: by completeness of the real numbers, the set $\{x \in [a, b] \colon f(x) < m\}$ has a least upper bound, call it c, and by continuity $f(c) = m$. And we can continue to fill in detail, according to the level at which we are solving the problem.

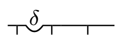
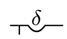
We should also bear in mind that not everything which fits under a microscope should be put under one, lest we get overwhelmed by the mass of minute information. As David Hume explains in his *Enquiry Concerning Human Understanding*, 'The abstractedness of these speculations is no recommendation, but rather a disadvantage to them. This difficulty may perhaps be surmounted by care and art, and the avoiding of all unnecessary detail.'

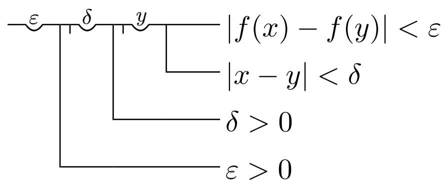
Finally, here is the definition of continuity in Frege's concept writing:



Here,  means ‘for every ε’,  means ‘B implies A’, and  A means ‘not A’.

Thus  means ‘A and B’ (it is not the case that B implies not A), and  means ‘there exists a δ’ (it is not the case that there is no δ).

We can cancel the double negative and simplify  to . This yields



Kiril Datchev, February 19, 2024. Questions, comments, and corrections are gratefully received at kdatchev@purdue.edu. The Frege quotes are adapted from Bauer-Mengelberg's and Bynum's translations.