LECTURE NOTES
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Linear Independence, Span, and Basis of a Set of Vectors

What is linear independence?

A set of vectors $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is **linearly independent** if none of the vectors $\mathbf{v}_i$ can be written as a linear combination of the other vectors, i.e. $\mathbf{v}_j = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$.

Suppose the vector $\mathbf{v}_j$ *can* be written as a linear combination of the other vectors, i.e. there exist scalars α_i such that $\mathbf{v}_j = \alpha_1\mathbf{v}_1 + \cdots + \alpha_k\mathbf{v}_k$ holds. (This is equivalent to saying that the vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly *dependent*). We can subtract $\mathbf{v}_j$ to move it over to the other side to get an expression $0 = \alpha_1\mathbf{v}_1 + \cdots \alpha_k\mathbf{v}_k$ (where the term $\mathbf{v}_j$ now appears on the right hand side.

In other words, the condition that "the set of vectors $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly *dependent*" is equivalent to the condition that there exists α_i not all of which are zero such that

$$0 = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k\end{bmatrix}\begin{bmatrix}\alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k\end{bmatrix}.$$

More concisely, form the matrix $\boldsymbol{V}$ whose columns are the vectors $\mathbf{v}_i$. Then the set S of vectors $\mathbf{v}_i$ is a linearly dependent set if there is a nonzero solution $\mathbf{x}$ such that $\boldsymbol{V}\mathbf{x} = 0$.

This means that the condition that "the set of vectors $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly *independent*" is equivalent to the condition that "the only solution $\mathbf{x}$ to the equation $\boldsymbol{V}\mathbf{x} = 0$ is the zero vector, i.e. $\mathbf{x} = 0$.

How do you determine if a set is lin. ind.?

To determine if a set $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ is linearly independent, we have to determine if the equation $\boldsymbol{V}\mathbf{x} = 0$ has solutions other than $\mathbf{x} = 0$. To do this,

1. Form the matrix $\boldsymbol{V}$ whose columns are the vectors $\mathbf{v}_i$.
2. Put $\boldsymbol{V}$ in row echelon form. Denote the row echelon form of $\boldsymbol{V}$ by $\texttt{ref}(\boldsymbol{V})$
3. check if each column contains a leading 1.

If every column of $\texttt{ref}(\boldsymbol{V})$ contains a leading 1, then $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_k\}$ **is linearly independent**. Otherwise, the set S is linearly *dependent*.

Example: Let $V = \mathbb{R}^4$, and let $T = \left\{ \begin{bmatrix}1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}3\\1\\0\\1\end{bmatrix}, \begin{bmatrix}-2\\1\\2\\-1\end{bmatrix} \right\}$. Is T linearly independent?

Answer To answer this, we do the following:

1. Form a matrix whose columns are the vectors in T. Call the matrix $\boldsymbol{M}_T$.
2. Row reduce $\boldsymbol{T}$ until it is in row echelon form, $\texttt{ref}(\boldsymbol{M}_T)$.
3. Check if each column has a leading 1.

Step 1. Form a matrix $\boldsymbol{M}_T$ whose columns are the vectors in the set T:

$$\begin{bmatrix}1\\0\\2\\0\end{bmatrix} \begin{bmatrix}3\\1\\0\\1\end{bmatrix} \begin{bmatrix}-2\\1\\2\\-1\end{bmatrix} \to \boldsymbol{M}_T = \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\2 & 0 & 2\\0 & 1 & -1\end{bmatrix}$$

Step 2. Row reduce the matrix $\boldsymbol{M}_T$.

$$\begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\2 & 0 & 2\\0 & 1 & -1\end{bmatrix} \quad R_3 \to R_3 - 2R_1 \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & -6 & 6\\0 & 1 & -1\end{bmatrix} \quad R_3 \to R_3 + 6R_4 \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & 0 & 0\\0 & 1 & -1\end{bmatrix}$$

$$\begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & 0 & 0\\0 & 1 & -1\end{bmatrix} \quad R_3 \leftrightarrow R_4 \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & 1 & -1\\0 & 0 & 0\end{bmatrix} \quad R_3 \to R_3 - R_2 \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & 0 & -2\\0 & 0 & 0\end{bmatrix}$$

$$R_3 \to \frac{1}{-2}R_3 \begin{bmatrix}1 & 3 & -2\\0 & 1 & 1\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

and now we can stop because we've reached row echelon form.

Step 3. What does this tell us? **Because the row echelon form has a "leading 1" in each column, the columns of the original matrix are *linear independent*.** This also tells us the vectors in our original set T are also linearly independent.

On the other hand, if any columns of the row echelon form did *not* contain a leading 1, then the original column vectors would then be linear *de*pendent.

Determining if a set of vectors spans a vectorspace

A set of vectors $F = \{\mathbf{f}_1, \cdots, \mathbf{f}_n\}$ taken from a vectorspace V is said to **span** the vectorspace if every vector in the vectorspace V can be expressed as a linear combination of the elements in F. In other words, every vector $\mathbf{x}$ in V can be written $\mathbf{x} = y_1\mathbf{f}_1 + \cdots + y_n\mathbf{f}_n$ for some scalars y_j. We can rewrite this idea from a matrix perspective:

$$\mathbf{x} = y_1\mathbf{f}_1 + \cdots + y_n\mathbf{f}_n = \begin{bmatrix}\mathbf{f}_1 & \cdots & \mathbf{f}_n\end{bmatrix} \begin{bmatrix}y_1\\ \vdots \\ y_n\end{bmatrix} \tag{1}$$

This matrix approach leads us to the method we use to determine whether our set of vectors F spans the vectorspace V. Let's be more concrete. The vectorspaces we deal with in this class tend to be like $\mathbb{R}^n$ – the set of vectors with n entries that are any real numbers. To show that the set F spans the vectorspace $\mathbb{R}^n$, we do the following:

0. Form the matrix $\boldsymbol{F} = \begin{bmatrix}\mathbf{f}_1 & \cdots & \mathbf{f}_n\end{bmatrix}$ with the vectors $\mathbf{f}_j$ as its columns
1. Compute the reduced row echelon form of that matrix $\boldsymbol{F}$, $\text{rref}(\boldsymbol{F})$.

2. If rref($\boldsymbol{F}$) has a leading 1 in every **row**, then the set F spans the vectorspace $\mathbb{R}^n$!

Determining if a set of vectors is a basis for a vectorspace

A **basis** for a vectorspace V is a set of vectors $B = \{\mathbf{b}_1, \cdots, \mathbf{b}_m\}$ that (1) span the vectorspace B; and (2) are linearly independent.

To determine if a set $B = \{\mathbf{b}_1, \cdots, \mathbf{b}_m\}$ of vectors spans V, do the following:

0. Form the matrix $\boldsymbol{B} = \begin{bmatrix}\mathbf{b}_1 & \cdots & \mathbf{b}_m\end{bmatrix}$
1. Compute rref($\boldsymbol{B}$)
2. Test for linear independence: does every column of rref($\boldsymbol{B}$) have a leading 1? (if yes, the set B is linearly independent)
3. Test whether B spans the vectorspace: does every row of rref($\boldsymbol{B}$) have a leading 1? (If yes, then the set B spans the vectorspace).
4. If $\boldsymbol{B}$ passes both tests, then the set B is a basis!

Determining a linearly independent subset of a set of vectors

Suppose we find out that the set of vectors $G = \{\mathbf{g}_1, \cdots, \mathbf{g}_k\}$ spans the vectorspace $\mathbb{R}^m$, but the set G is not linearly independent. How can we find a subset of G that *is* linearly independent? In other words, can we find a basis for our vectorspace $\mathbb{R}^m$ hidden inside our linearly dependent set of vectors G?

Do the following:

0. As always, first form a matrix $\boldsymbol{G} = \begin{bmatrix}\mathbf{g}_1 & \cdots & \mathbf{g}_k\end{bmatrix}$
1. Then compute rref($\boldsymbol{G}$).
2. Each column of rref($\boldsymbol{G}$) that contains a leading 1 corresponds to a vector $\mathbf{g}_j$ in the original set G. Let S be the subset of those vectors. Then S is linearly independent, **AND span**(S) = **span**(G)**.** This means that S is a basis for the span of G !!

Example: Let $V = \mathbb{R}^3$, and let $W = \left\{ \begin{bmatrix}2\\4\\-4\end{bmatrix}, \begin{bmatrix}2\\1\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\8\end{bmatrix}, \begin{bmatrix}-2\\6\\6\end{bmatrix} \right\}$. Find a subset of W that is a basis for V.

Step 0. First form the matrix $\boldsymbol{W} = \begin{bmatrix}2 & 2 & 0 & -2\\4 & 1 & 1 & 6\\-4 & -2 & 8 & 6\end{bmatrix}$.

Step 1. Compute rref($\boldsymbol{W}$) $= \begin{bmatrix}1 & 0 & 0 & 2\\0 & 1 & 0 & -3\\0 & 0 & 1 & 1\end{bmatrix}$.

Step 2. First note that not every column contains a leading 1 – that means that our original set T is not linearly independent, and so it cannot be a basis.

However, the first three columns of rref($\boldsymbol{W}$) contain a leading 1. If we let $S = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ (since those are the three vectors that correspond to the columns of rref($\boldsymbol{W}$) that contain leading 1s), then S is a linearly independent set. Since each row of rref($\boldsymbol{W}$) contains a leading 1, we know that W spans the vectorspace. But the columns of rref($\boldsymbol{W}$) that correspond to our subset of vectors, S, also all contain leading 1s (our subset S is the first three vectors, $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$; this corresponds to the first three columns of rref($\boldsymbol{W}$)) – that means that our subset S still spans the vectorspace!

1 Determining a basis for span(S) without using vectors from S

We have seen already that you can locate a linearly independent set of vectors within the set of vectors $S = \{\mathbf{s}_1, \cdots, \mathbf{s}_m\}$ by forming a matrix $\boldsymbol{S} = \begin{bmatrix}\mathbf{s}_1 & \cdots & \mathbf{s}_m\end{bmatrix}$, computing rref($\boldsymbol{S}$), and then taking each of the vectors $\mathbf{s}_j$ that corresponds to a column of rref($\boldsymbol{S}$) that contains a leading 1.

In lessons 22-23 (class on 10/20, 10/22) we'll look at an examples of finding a basis for S using things other than vectors taken directly from S.