LinGloss

A glossary of linear algebra

Contents:

Quasi-triangular A matrix $\mathbf{A}$ is quasi-triangular iff it is a triangular matrix except its diagonal is composed of 1×1 and 2×2 blocks.

Gram Matrix (**Gramian Matrix**, **Gramian**)

(of a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_n$) is the matrix $\mathbf{G}$ such that $\mathbf{G}_{ij} = \mathbf{v}_i \cdot \mathbf{v}_j$. The set is linearly independent iff $\det(\mathbf{G}) \neq 0$

Cramer's Rule For $\mathbf{A} \in \mathbb{F}^{n \times n}$ we can compute $\mathbf{x} : \mathbf{Ax} = \mathbf{b}$ by

$$x_i = \frac{\det(\mathbf{A}(i)}{\det(\mathbf{A})}$$

where $\mathbf{A}(i) := \{\mathbf{A} \text{ with } \mathbf{b} \text{ replacing the } i^{\text{th}} \text{ column } \}$.

Bilinear form For a vectorspace $\mathbf{V}$ over a field $\mathbb{K}$, $f : \mathbf{V} \times \mathbf{V} \to \mathbb{K}$ is a bilinear form iff it's linear in each component:

$$f(a\mathbf{v}_1 + b\mathbf{v}_2, \mathbf{u}) = af(\mathbf{v}_1, \mathbf{u}) + bf(\mathbf{v}_2, \mathbf{u})$$

- for all $\mathbf{A} \in \mathbb{F}^{n \times n}$, $f_A(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x}$ defines a bilinear form, f_A.
- Bilinear forms on n-dimension $\mathbf{V}$ are in 1-1 correspondence with $\mathbf{A} \in \mathbb{K}^{n \times n}$.
- f is **symmetric** iff $\mathbf{A}_f$ is symmetric, iff $f(\mathbf{v}, \mathbf{u}) = f(\mathbf{u}, \mathbf{v})$.
- f is **positive definite** iff $\mathbf{A}_f$ is positive definite.
- f is **non-degenerate** iff $\text{rank}(f) = \text{rank}(\mathbf{V})$.

Inner Product Space (**Euclidean Space**)

A vectorspace V equipped with an inner product $\langle \cdot, \cdot \rangle$ satsifying for all $\mathbf{v}, \mathbf{u} \in V$

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2. $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ iff $\mathbf{v} = 0$
3. $\langle \cdot, \cdot \rangle$ is bilinear.

Equivalent matrices $\mathbf{A}$ and $\mathbf{B}$ are equivalent iff $\mathbf{A} = \mathbf{Q}^{-1}\mathbf{BP}$ for invertible $\mathbf{Q}$, $\mathbf{P}$. For rectangular matrices, "equivalence" as matrices means the matrices represent the same linear transformation.

Similar matrices $\mathbf{A}$ and $\mathbf{B}$ are similar iff $\mathbf{A} = \mathbf{P}^{-1}\mathbf{BP}$ for invertible $\mathbf{P}$. Note that this is a relation on square matrices only.

Moore-Penrose Inverse (**Pseudo-inverse**, **generalized inverse**) Given $\mathbf{A} \in \mathbb{F}^{m\times n}$ there exists a matrix $\mathbf{A}^+$ which satisfies:

$$\mathbf{A}\mathbf{A}^+\mathbf{A} = \mathbf{A} \tag{1}$$
$$\mathbf{A}^+\mathbf{A}\mathbf{A}^+ = \mathbf{A}^+ \tag{2}$$
$$\mathbf{A}\mathbf{A}^+ = \left(\mathbf{A}\mathbf{A}^+\right)^T \tag{3}$$
$$\mathbf{A}^+\mathbf{A} = \left(\mathbf{A}^+\mathbf{A}\right)^T \tag{4}$$
$$\tag{5}$$

- $\mathbf{A}^+$ is unique
- if $r = \text{rank}\mathbf{A}$ and we factor $\mathbf{A} = \mathbf{B}\mathbf{C}$ having dimensions $m \times r$, $r \times, n$, then we have

$$\mathbf{A}^+ = \mathbf{C}^T \left(\mathbf{C}\mathbf{C}^T\right)^{-1} \left(\mathbf{B}^T\mathbf{B}\right)^{-1} \mathbf{B}^T$$

Group Inverse $\mathbf{A} \in \mathbb{F}^{n\times n}$ can have a group inverse $\mathbf{A}^\# \in \mathbb{F}^{n\times n}$ satisfying

$$\mathbf{A}\mathbf{A}^\# = \mathbf{A}^\#\mathbf{A} \tag{6}$$
$$\mathbf{A}^\#\mathbf{A}\mathbf{A}^\# = \mathbf{A}^\# \tag{7}$$
$$\mathbf{A}\mathbf{A}^\#\mathbf{A} = \mathbf{A} \tag{8}$$
$$\tag{9}$$

- if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda \neq 0$, then $\mathbf{v}$ is an eigenvector of $\mathbf{A}^\#$ with eigenvalue $\frac{1}{\lambda}$
- if $\mathbf{v}$ is an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda = 0$, then $\mathbf{v}$ is an eigenvector of $\mathbf{A}^\#$ with eigenvalue 0
- $\mathbf{A}^\#$ is unique if it exists. If $\mathbf{A}^{-1}$ exists, then $\mathbf{A}^{-1} = \mathbf{A}^\#$.
- Let $\mathbf{A} = \mathbf{B}\mathbf{C}$ have dimensions $n \times r$, $r \times n$. If $\mathbf{C}\mathbf{B}$ is invertible, then $\mathbf{A}^\#$ exists and $\mathbf{A}^\# = \mathbf{B}\left(\mathbf{C}\mathbf{B}\right)^{-2}\mathbf{C}$

Spectral radius If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$ (possibly real) then the spectral radius is

$$\rho\left(\mathbf{A}\right) := \max_i\{|\lambda_i|\}$$

Note: $\rho(\mathbf{A}) \leq \|\mathbf{A}^k\|^{1/k}$ for all $k \in \mathbb{N}$ for any consistent matrix norm $\|\cdot\|$

Matrix Norm A matrix norm $\|\cdot\|$ is a map $\|\cdot\| : \mathbb{R}^{n\times m} \to \mathbb{R}$ satisfying

(1) $\|\mathbf{A}\| \geq 0$ for all $\mathbf{A} \in \mathbb{R}^{n\times m}$, $\|\mathbf{A}\| = 0$ iff $\mathbf{A} = 0$

(2) $\|\mathbf{A} + \mathbf{B}\| \leq \|\mathbf{A}\| + \|\mathbf{B}\|$ for all $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n\times m}$

(3) $\|\alpha\mathbf{A}\| = |\alpha|\|\mathbf{A}\|$ for all $\alpha \in \mathbb{R}$ and $\mathbf{A} \in \mathbb{R}^{n\times m}$

Matrix norms $\|\cdot\|_a, \|\cdot\|_b, \|\cdot\|_c$ on appropriate spaces are **consistent** iff $\|\mathbf{A}\mathbf{B}\|_a \leq \|\mathbf{A}\|_b\|\mathbf{B}\|_c$ always holds.

We can define a matrix norm $\|\cdot\|_{a,b}$ that is **subordinate** to the vector norms $\|\cdot\|_a$, $\|\cdot\|_b$ by

$$\|\mathbf{A}\|_{a,b} := \sup_{\mathbf{x}\neq 0} \frac{\|\mathbf{Ax}\|_b}{\|\mathbf{x}\|_a}$$

Note that subordinate matrix norms satisfy

$$\|\mathbf{A}\|_{a,b} := \max_{\|\mathbf{x}\|_a=1} \frac{\|\mathbf{Ax}\|_b}{=} \|\mathbf{Ax}^*\|_b$$

Frobenius Norm For $\mathbf{A} \in \mathbb{K}^{m\times n}$ we have $\|\mathbf{A}\|_F = \sqrt{\sum_i \sum_j |\mathbf{A}_{ij}|^2}$

$= \sqrt{\operatorname{tr}(\mathbf{AA}^*)}$

$= \sqrt{\sum_i \sigma_i^2}$, where σ_i are the singular values of $\mathbf{A}$

Compare with $\|\mathbf{A}\|_2^2 = \sigma_1^2$ (and in the case $\mathbf{A}$ is square, $\sigma_1^2 = \lambda_1^2$).

Hermitian Matrix (**Self-adjoint**)

$\mathbf{M} \in \mathbb{F}^{n\times n}$ is Hermitian iff $\mathbf{M}$ equals its Hermitian transpose (conjugate transpose, Hermitian conjugate, Adjoint Matrix), i.e. $\mathbf{M} = \mathbf{M}^* = \overline{\mathbf{M}}^T = \mathbf{M}^\dagger = \mathbf{M}^H$, i.e. $\mathbf{M}_{ij} = \overline{\mathbf{M}}_{ji}$. Note: Hermitian implies normal.

Projection Matrix (**Projection, Idempotent Matrix, Projector**)

$\mathbf{A}$ is a projection matrix iff $\mathbf{A}^2 = \mathbf{A}$.
- all eigenvalues are 0 or 1
- the minimimum polynomial divides $x^2 - x$.
- $\det \mathbf{A} = 0$ or 1.

Markov Matrix (**Stochastic Matrix**)

$\mathbf{M} \in \mathbb{R}^{n\times n}$ is Markov iff for all i, j

(1) $0 \leq M_{ij} \leq 1$

(2) $\sum_{j=1}^{n} M_{ij} = 1$

- for Markov $\mathbf{M}, \mathbf{N}$; $\mathbf{MN}$ is also Markov.
- "Column stochastic" means this same property, with respect to columns.

Orthogonal Matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$ is orthongal iff $\mathbf{M}^T\mathbf{M} = \mathbf{MM}^T = \mathbf{I}$.
- $|\det \mathbf{M}| = 1$
- preserves inner-products ($< \mathbf{v}, \mathbf{w} >=< \mathbf{Av}, \mathbf{Aw} >$)
- preserves lengths, i.e. is an isometry
- eigenvalues INCOMPLETE

Householder Matrix (**Householder Reflector, Reflector Matrix**)

For any $\mathbf{u} \in \mathbb{F}^n$ with $\mathbf{u}^T\mathbf{u} = 1$, the Householder reflector of $\mathbf{u}$ is $\mathbf{H}_\mathbf{u} = \mathbf{I} - 2\mathbf{uu}^T$.
- symmetric
- orthogonal
- eigenvalues: -1 with multiplicity 1, 1 with multiplicity $n-1$.

- eigenvectors: **u** with multiplicity 1; the other $n-1$ are the $n-1$ basis vectors of null(**u**)

Unitary Matrix $\mathbf{A} \in \mathbb{C}^{n \times n}$ is unitary iff $\mathbf{A}^*\mathbf{A} = \mathbf{A}\mathbf{A}^* = \mathbf{I}$.

- $|\det(\mathbf{A})| = 1$
- preserves inner-products ($<\mathbf{v}, \mathbf{w}> = <\mathbf{A}\mathbf{v}, \mathbf{A}\mathbf{w}>$)
- preserves lengths, i.e. is an isometry
- eigenvalues INCOMPLETE

Congruent Matrices Two matrices $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{n \times n}$ are congruent iff $\mathbf{A} = \mathbf{P}^T\mathbf{B}\mathbf{P}$ for some invertible matrix **P** (note: **P** need not be orthogonal!)

- congruent matrices represent the same bilinear form with respect to different bases
- different from similar matrices, which represent the same linear transformation with respect to different bases
- different from orthogonally-similar matrices, which represent rotations of the same linear transformation
- Law of Inertia: congruent matrices have the same number of positive, negative, and zero eigenvalues (this means they have the same signature as well)

Deficient Matrix (**Defective Matrix**)

A matrix with less than n linearly independent eigenvectors, i.e. at least one eigenvalue has a Jordan block of size > 1. A matrix is deficient iff it is non-diagonalizable.

Derogatory Matrix A matrix is derogatory if an eigenvalue has more than one Jordan block associated to it. Note that this is distinct from simply having a repeated eigenvalue – a deficient matrix has at least one repeated eigenvalue but without a corresponding number of eigenvectors; that is, the eigenvalue has algebraic multiplicity $a_\lambda > 1$, but geometric multiplicity $g_\lambda < a_\lambda$. A derogatory matrix, on the other hand, simply has $a_\lambda > 1$ for at least one λ – it is not necessarily the case that g_λ equals a_λ or not.

- non-derogatory iff characteristic polynomial = minimal polynomial.

Simple eigenvalue An eigenvalue with algebraic multiplicity = 1 (which necessarily implies geometric multiplicity = 1).

Semisimple eigenvalue An eigenvalue for which all associated Jordan blocks have dimension 1; i.e. algebraic multiplicity = geometric multiplicity.

Defective eigenvalue (**Deficient eigenvalue**)

An eigenvalue that is not semisimple: an eigenvalue with geometric multiplicity ¡ algebraic multiplicity. A matrix has a defective eigenvalue iff it is a defective matrix (a deficient matrix).

Signature (of a symmetric bilinear form)

Any symmetric bilear form on a finite dimensional vectorspace has a diagonal matrix representation. The signature is

$s = p - n$

# of positive entries on the diagonal $-$# of negative entries on the diagonal

Isometry A linear transformation $\mathbf{T}$ on $\mathbb{F}$ is an isometry iff $\|\mathbf{Tv}\| = \|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbb{F}^n$.

- an isometry is necessarily invertible
- all eigenvalues satisfy $|\lambda| = 1$.

Rank (of a matrix): For $\mathbf{M} \in \mathbb{F}^{n\times n}$, rank$\mathbf{M}$:= # of independent columns (= # of independent rows)

- if $\mathbf{A} \in \mathbb{F}^{m\times m}$ and $\mathbf{C} \in \mathbb{F}^{n\times n}$ are nonsingular then

$$\text{rank}(\mathbf{AM}) = \text{rank}(\mathbf{MC}) = \text{rank}(\mathbf{M})$$

- for $\mathbf{A}, \mathbf{B} \in \mathbb{F}^{n\times n}$, $\mathbf{AB}$ nonsingular iff $\mathbf{A}$ and $\mathbf{B}$ are nonsingular.
- rank($\mathbf{M}$) = dimension of the largest nonsingular submatrix of $\mathbf{M}$.
- For arbitrary $\mathbf{U}, \mathbf{V}$ we have $0 \leq \text{rank}(\mathbf{UV}) \leq \min(\text{rank}(\mathbf{U}), \text{rank}(\mathbf{V}))$
- It is possible $\mathbf{UV} = 0$, even if one of them is non-singular: $\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Normal Matrix $\mathbf{A} \in \mathbb{F}^{n\times n}$ is normal iff $\mathbf{A}^*\mathbf{A} = \mathbf{A}^*\mathbf{A}$.

Theorem: normal iff unitarily diagonalizable, meaning $\mathbf{QAQ}^* = \mathbf{D}$ for unitary $\mathbf{Q}$ and diagonal $\mathbf{D}$.

- note: being diagonalizable does not imply unitarily diagonalizable.
- for real matrices, being orthogonally diagonalizable implies symmetric: $\mathbf{A} = \mathbf{QDQ}^T = \left(\mathbf{QDQ}^T\right)^T$, and being symmetric implies normal, so for $\mathbb{R}$, SYMMETRIC = ORTHOGONALLY DIAGONALIZABLE = NORMAL

1 Theorems

Spectral Mapping Theorem Let $\mathbf{A} \in \mathbb{F}^{n\times n}$ have eigenvalues $\lambda_1, \cdots, \lambda_n$ not necessarily distinct, and take $p(x) \in \mathbb{F}[x]$. Then the eigenvalues of $p(\mathbf{A})$ are $p(\lambda_1), \cdots, p(\lambda_n)$. Note the characteristic polynomial has changed from $\prod(x-\lambda_i)$ to $\prod(x - p(\lambda_i))$

Perron-Frobenius Theorem If $\mathbf{A} \in \mathbb{F}^{n\times n}$ has all positive entries then

1. There exists a unique eigenvector with all positive entries (up to scalar multiples)
2. Its corresponding eigenvalue is λ_1
3. That eigenvalue has multiplicty 1 (i.e. it is a simple eigenvalue)

Schur Decomposition For any $\mathbf{A} \in \mathbb{C}^{n \times n}$ there exists a unitary $\mathbf{Q} \in \mathbb{C}^{n \times n}$ and upper triangular $\mathbf{U} \in \mathbb{C}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{QUQ}^*$$

- not necessarily unique

- an extension of the spectral decomposition theorem (i.e. diagonalization theorem)

- If $\mathbf{A}$ is positive definite, then the Schur, Singular Value, and Spectral decompositions are the same.

- if $\mathbf{A}$ is $\mathbb{R}$, then there exists orthogonal $\mathbf{Q} \in \mathbb{R}^{n \times n}$ and upper quasi-triangular $\mathbf{T} \in \mathbb{R}^{n \times n}$ such that $\mathbf{A} = \mathbf{QTQ}^T$.

Singular Value Decomposition (SVD) Any $\mathbf{M} \in \mathbb{C}^{n \times m}$ has a Singular Value Decomposition $\mathbf{M} = \mathbf{U}\Sigma\mathbf{V}^*$ for $\mathbf{U}$ m×m unitary, $\mathbf{V}$ n×n unitary, and Σ m×n diagonal with the singular values as its entries.

- $\mathbf{U}$ and $\mathbf{V}$ aren't unique, but Σ is up to re-ordering.

- If $\mathbf{M}$ is diagonalizable