Example. Use Maclaurin Series to show that $e^{ix} = \cos x + i \sin x$, where $i^2 = -1$.

Solution: Recall the Maclaurin series for e^x, $\cos x$ and $\sin x$:

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} && = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \frac{x^7}{7!} + \cdots \\ \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} && = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} && = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}. \end{aligned}$$

Using the series for e^x, we get

$$\begin{aligned} e^{ix} &= 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} + \cdots \\ &= 1 + ix + i^2\frac{x^2}{2!} + i^3\frac{x^3}{3!} + i^4\frac{x^4}{4!} + i^5\frac{x^5}{5!} + i^6\frac{x^6}{6!} + i^7\frac{x^7}{7!} + \cdots \end{aligned}$$

Notice that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, and the pattern repeats from there. So

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} - i\frac{x^7}{7!} + \cdots$$

Now group by terms with an i (the odd terms) and terms without an i (the even terms):

$$\begin{aligned} e^{ix} &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + ix - i\frac{x^3}{3!} + i\frac{x^5}{5!} - i\frac{x^7}{7!} + \cdots \\ &= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) \end{aligned}$$

Finally, we can recognize the first term in the parentheses as $\cos x$, and the second set of parentheses as $\sin x$. In conclusion,

$$e^{ix} = \cos x + i \sin x.$$

Neat fact: now you can talk about exponentials of complex numbers. For example,

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 \text{ and } e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2} = i.$$

This identity will be used later in the semester when we talk about second order linear differential equations with complex eigenvalues.